A Two-Sided Matching Model of Monitored Finance*

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Abstract

We analyze an incentive contracting model of partnership formation between heterogeneous investors and entrepreneurs. Partnerships are subject to double-sided moral hazard problems in entrepreneurial action and monitoring by investors. Greater monitoring ability implies stronger incentives to monitor. On the other hand, low-collateral borrowers have lower inside equity participation. Hence, the incentive problem is best mitigated by assigning low-collateralized entrepreneurs to high-ability investors following a negative assortative matching pattern. Moreover, negative assortative matching implies that the equilibrium loan rate is in general non-monotonic in borrower collateral. Finally, our model sheds light on how changes in the inequality of collateral distribution affect the cost of external borrowing.

JEL codes: C78, D86, G23.
Key words: Lender monitoring; assortative matching; loan contracts.

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Introduction

It is well-known that borrower collateral plays an important role in ameliorating incentive problems in a borrower-lending relationship. Notwithstanding this, there is a lack of consensus on the role of collateral in determining the essential characteristics of a loan contract such as the loan rate. Ex post theories of collateral assert that observably riskier borrowers, who pay a higher loan rate, are often required to pledge higher collateral to reduce agency costs (hidden action), and hence, there is a positive association between the loan rate and collateral (e.g. Boot and Thakor, 1994). Ex ante theories of collateral, by contrast, postulate that when borrower quality is unobservable (hidden information), safer borrowers tend to pledge higher collateral to signal quality, and hence, there is a negative association (e.g. Besanko and Thakor, 1987). Empirical findings endorse both views.[1] Berger, Frame, and Ioannidou (2016) find empirical evidence of a non-monotonic relation between loan rate and collateral.

Given the aforementioned dissent on the theoretical predictions regarding the optimal association between loan rate and collateral, our paper aims at providing a unified framework which is amenable to rationalize a possible non-monotonicity of loan rate with respect to borrower collateral. For that purpose, we analyze a competitive credit market in which entrepreneurs form partnerships with investors. Risk-neutral entrepreneurs are heterogeneous with respect to initial wealth which can be fully pledged as collateral. Borrower wealth is not sufficient to cover the fixed project cost, and hence, all entrepreneurs must rely on external borrowing. A loan contract, which specifies the loan rate to be paid to the lender, is subject to limited liability. Borrower moral hazard stems from the fact that an entrepreneur, in the absence of any monitoring, may deliberately decrease the probability of obtaining a high cash flow of his project in order to consume private benefits (as in Hölmstrom and Tirole, 1997). Lenders can potentially mitigate the borrower moral hazard problem by costly monitoring. Risk-neutral investors are heterogeneous with respect to their monitoring ability. Lenders with a greater monitoring ability are the ones who entail a lower marginal cost of monitoring, and hence, are more efficient. However, an investor cannot credibly commit to a predetermined level of monitoring, which gives rise to a lender moral hazard problem. Therefore, endogenous partnership formation is subject to a double-sided moral hazard problem that hinders the implementation of efficient outcomes.

Differences in monitoring ability implies differences in the lender moral hazard problem as more efficient investors have stronger incentives to monitor. Because monitoring enhances firm value, competition for more efficient lenders naturally emerges in such market. To capture this idea, we model the lender-borrower market as a two-sided assignment game (e.g. Shapley and Shubik, 1971). Investors with greater monitoring ability have comparative advantage in lending to low-collateral firms. Thus, to maximize efficiency in each partnership, it is optimal to assign low-collateral firms to high-ability in-

[1] See Berger and Udell (1990); Brick and Palia (2007) for a positive relationship and Degryse and Van Cayseele (2000); Agarwal and Hauswald (2010) for a negative relationship between loan rate and borrower collateral. Boot, Thakor, and Udell (1991) analyze a model of secured lending by combining the above two attributes – hidden action and hidden information – and show that default risk, and hence, loan rate may be either increasing or decreasing with respect to borrower quality.

vestors following a *negative assortative matching* (NAM) pattern. In other words, monitoring ability and collateral are substitutes in mitigating the associated double-sided moral hazard problem.

One main result of our model is a potential non-monotonic association between loan rate and collateral. The intuition is as follows. The optimal loan rate associated with an isolated lender-borrower partnership, which serves as the instrument that balances double-sided incentive problems, is in general a function of monitoring ability (lender type), collateral (borrower type), and the exogenous outside option of the borrower. An investor with lower monitoring ability (i.e., higher marginal cost of monitoring) has weaker incentives to monitor, and hence, requires higher marginal compensation (in the form of loan rates) to exert an additional unit of monitoring effort. By contrast, a borrower with high collateral requires less intense monitoring, and hence, the monitor must retain a smaller portion of the realized cash flow. Finally, if the exogenous outside option of an entrepreneur increases, then he must pay a lower loan rate due to the increased bargaining power. To understand why under endogenous sorting the equilibrium loan rate may be non-monotonic in borrower collateral, consider two investor-entrepreneur partnerships with two distinct levels of collateral and monitoring ability. First, note that in an assignment model, each borrower's outside option is endogenous, and is increasing in borrower type. In equilibrium, the borrower with high collateral is matched with a less efficient monitor following NAM. Thus, this entrepreneur must pay a lower loan rate than the other borrower because both his collateral and outside option are higher. However, NAM implies that the matched partner of this high-collateral entrepreneur must receive a higher loan rate because she has weaker incentives to monitor. Because of these countervailing effects, the equilibrium loan rate may be non-monotonic in collateral.

As is typical with assignment models, the nature of equilibrium matching (NAM in this particular context) between investors and entrepreneurs is independent of the type distributions. However, the shapes of the equilibrium matching and payoff functions may change following a change in the type distributions. Terviö (2008) shows how changes in type distributions induce the equilibrium matching function to change, and has positive spillover effects on the upper tail of the type distributions in terms of equilibrium payoffs. In our model, changes in type distributions have important comparative statics implications for the equilibrium loan rate as a function of collateral. If, at a given level of collateral, the number of borrowers increases relative to the number of lenders with the corresponding monitoring ability, then these lenders gain higher bargaining power because the borrowers are now relatively abundant. As a result, the same type of borrowers must pay a higher loan rate. By exploiting this simple intuition, we carry out a numerical exercise to show the effect of a more unequal distribution of collateral on the equilibrium matching function and the relative bargaining power of the borrowers. In particular, we consider two distributions of collateral based on yearly data of Italian firms, where one distribution is more unequal than the other (in the sense of second-order stochastic dominance). Such a cross-sectional variation in the collateral distribution affects the equilibrium loan rate and monitoring functions via the shifts in the matching function and bargaining power of the borrowers. The resultant shifts in the matching and borrower utility functions may not point in the same direction, and consequently, following a change in the collateral distribution, some borrowers pay higher loan rates, whereas others end up paying lower rates. The effect of such cross-sectional change in the distribution of collateral is also asymmetric with respect to the equilibrium monitoring intensity, and hence, the productive efficiency.

Our assumption about the differences in monitoring abilities across lenders is motivated by the works of Stein (2002) and Berger, Miller, Petersen, Rajan, and Stein (2005). Stein (2002) studies how different organizational structures can generate and process information on investment projects. In the context of banking, Stein's (2002) framework postulates that large banks have a disadvantage – relative to small banks – in collecting "soft" information that cannot be credibly transmitted. The prediction that small banks have a comparative advantage over large ones in processing soft information is confirmed empirically by Berger et al. (2005). Thus, the differences in monitoring ability in our model may be interpreted in terms of the differences across lenders in processing soft information.

The contribution of the present paper to the literature on incentive contracting and partnership formation is two-fold. First, when the individuals seek for alternative partners, the model helps endogenize the outside option of each borrower as opposed to the models in which the outside options of the individuals on one side of the market are exogenously given (e.g. Besanko and Kanatas, 1993; Hölmstrom and Tirole, 1997; Repullo and Suárez, 2000). The aforementioned models are amenable to determine the optimal incentive structure in an organization in the sense that they determine the way a fixed surplus must be divided between the lender and borrower in a given lending relationship. A fixed outside option of a borrower also pins down the payoff achievable by his matched partner. In an assignment model such as ours, the endogenous outside option not only determines the structure of incentive pay, but also its level in each partnership, and hence, the bargaining power of each individual is endogenous. In Section 4, we show how changes in type distributions alter the endogenous bargaining power of the market participants. Second, we contribute to the literature on partnership formation (e.g. Farrell and Sctochmer, 1988) which argues that economic agents who differ in abilities will form partnerships by equally sharing the surplus if abilities are complementary. In the context of corporate lending, formation of partnerships is often subject to several market imperfections, among which informational constraints play an important role. When partnerships are subject to moral hazard, incentive contract for a particular match gives rise to a non-linear Pareto frontier implying that match surplus cannot be transferred between the principal and agent on a one-to-one basis, and an equal sharing of surplus cannot be implemented. Thus, substitutability rather than complementarity explains why heterogeneous partnerships are formed, and individuals share the match output according to endogenously determined sharing rules.

1 Related literature

Legros and Newman (2007) extend the assignment game (e.g. Shapley and Shubik, 1971) to an environment with imperfect transferability where the Pareto frontier associated with each match is non-linear.

They propose the *generalized decreasing differences* (GDD) condition, which is a necessary and sufficient condition for NAM in markets with two-sided heterogeneity. Our model contributes to this strand of literature by showing that the double-sided moral hazard problem is a way to induce imperfect transferability in an assignment game. Moreover, we show that GDD holds in our model, and hence, the equilibrium allocation exhibits NAM. The present model resembles that of Chakraborty and Citanna (2005), which also analyzes partnerships under double-sided moral hazard. Because wealthier individuals are less wealth constrained, they accept occupations with more severe incentive problems. Partnerships are assortative, and an increase in median wealth improves the welfare of poorer agents.

In the context of [corporate] finance, two classes of papers analyze the effect of endogenous matching on incentive contracts. The first type studies the effects of endogenous investor-entrepreneur matching on optimal financial contracts.[2] Besley, Buchardi, and Ghatak (2012) analyze the effect of competition between lenders that are heterogeneous with respect to the cost of capital, and consider variations in property rights on optimal loan contracts. If competition is sufficiently intense (more similar lenders), the borrowers receive their outside option. Improved property rights, which allows the borrower to pledge a larger proportion of wealth as collateral, relaxes the borrower incentive problem and reduces the loan rate. However, Besley et al. (2012) consider matching markets with one-sided heterogeneity, and hence, do not take into account the effect of assortative matching.[3] Thus, any changes in market fundamentals affect equilibrium contracts only through the endogenous outside option. By contrast, we show that changes in type distributions may have asymmetric effects on the equilibrium allocations. One paper that considers the effects of lender-borrower sorting is that of Cabolis, Dai, and Serfes (2015), who analyze a venture capital (VC) market. They show positive assortative matching between VC rank (given by their stage-specific expertise) and firm quality (proxied by the probability of successful exit) at each stage. Moreover, those authors establish a non-monotonic relationship between specialization and competition.

2 The model

2.1 Lender-borrower matching

The economy, which spans three dates $t = 0, 1, 2$, consists of two classes of agents – a continuum $I = [0, 1]$ of risk-neutral investors (lenders) and a continuum $J = [0, 1]$ of risk-neutral entrepreneurs (borrowers). Entrepreneurs are heterogeneous with respect to their initial wealth. In particular, a type

[2]The second class of papers analyzes the effect of assigning managerial talent to firm characteristics on the optimal managerial compensation. There is assortative matching, i.e., more talented managers run larger firms (e.g. Terviö, 2008; Edmans, Gabaix, and Landier, 2009), more profitable firms (e.g. Alonso-Paulí and Pérez-Castrillo, 2012), safer firms (e.g. Li and Ueda, 2012), or firms with greater market power (e.g. Dam, 2015). Many of these papers show that assortative matching significantly explains the observed variations in the level and incentive structure of CEO pay.

[3]Dam and Pérez-Castrillo (2006) and von Lilienfeld-Toal and Mookherjee (2016) are the two other papers that consider the effect of endogenous matching on principal-agent contracts under one-sided heterogeneity.

w entrepreneur has initial wealth that can entirely be pledged as collateral whose market value is $w \in W = [w_{min}, w_{max}]$ with $w_{min} \geq 0$. Let $F(w)$ denote the fraction of borrowers with collateral lower than or equal to w. In other words, $F(w)$ is the cumulative distribution function of collateral, and let $f(w)$ be the corresponding density function with $f(w) > 0$ for all $w \in W$. Further, each entrepreneur has a startup project whose initial outlay is \$1. We assume that $w_{max} \leq 1$ so that all entrepreneurs require external funds to cover the project cost. A dollar invested in the project at date 1 yields a stochastic but verifiable cash flow $Q > 1$ (success) or 0 (failure) at $t = 2$.

A borrower can choose between two actions – "behave" (H) and "shirk" (L) which determine the probability of success of the project. Formally, let Prob.$\{Q \mid H\} = p_H$ and Prob.$\{Q \mid L\} = p_L$. Without loss of generality, we assume that $p_H = p \in (0, 1)$ and $p_L = 0$. Further, if a borrower misbehaves, she enjoys a private benefit $B > 0$. We assume $pQ > B$ so that the project is economically viable at least at the first-best level. By monitoring at intensity $m \in [0, 1]$, a lender can oblige an entrepreneur to behave with probability m, for which he has to incur a cost which takes the following functional form:[4]

$$D(m; c) = \frac{m^2}{2c}.$$

The parameter c represents the 'ability' or 'efficiency' of a lender or the lender 'type'. The higher the c is, the greater the ability is, as an lender with higher c entails a lower cost for each additional unit of monitoring.[5] No lender can pre-commit to such actions, and hence, costly monitoring gives rise to a *lender moral hazard* problem. We assume that lenders are heterogeneous with respect to monitoring ability. Let $G(c)$ be the cumulative distribution function of monitoring ability, and $g(c)$ be the corresponding density function with $g(c) > 0$ for all $c \in C = [c_{min}, c_{max}]$ with $c_{min} > 0$. Lender types are also publicly observable, and the type distributions are taken as the primitives of our economy under consideration. Let $\xi = (G, F)$ denote a generic lender-borrower economy or market with two-sided heterogeneity.

On date 0, if a lender agrees to finance an entrepreneur's project, then a lender-borrower partnership or match forms. As types, not individual names, matter, a typical partnership will be denoted by (c, w). We treat partnership formation as an endogenous matching problem in which a lender with a given ability c is assigned to a borrower with a given level of collateral w. To this end, we extend Sattinger's (1979) "differential rents" model to an environment in which utility is not perfectly transferable. Formally, each partnership (c, w) forms via a one-to-one matching rule $\lambda : W \to C$, which assigns to each collateral level

[4] All our results hold under a general form of monitoring cost function $D(m; c)$ which is strictly increasing and convex in m with $D(0; c) = 0$, $D_{12}(m; c) < 0$ and $D_{112}/D_{12} \leq 1 \leq mD_{11}/D_1$.

[5] Monitoring ability may sometimes be difficult to quantify, but can be proxied by the types of institutional investors. For example, Almazan, Hartzell, and Starks (2005) claim that, compared with bank trust departments and insurance companies, investment advisors and investment companies in general entail lower costs of monitoring. Ability may also be proxied by investor attributes such as expertise (e.g. Almazan, 2002), experience (e.g. Sorensen, 2007), independence of organizational structure (e.g. Bottazzi, Da Rin, and Hellmann, 2008), or size of the human capital (e.g. Dimov and Shepherd, 2005), which may limit the intensity of investor monitoring. The monitoring cost function we propose may thus be viewed as a reduced-form function in which ability either represents one of the aforementioned investor attributes or has a strong positive association with one of them.

$w \in W$ a lender ability $\lambda(w) \in C$. Let λ^{-1} denote the inverse matching function. One of our main objectives is to determine the equilibrium matching pattern, i.e., which types of lenders and borrowers will form partnerships. The following definition describes a negative assortative matching (NAM) pattern.

Definition 1 (Negative assortative matching) *Lender-borrower matching is negatively assortative if $\lambda'(w) \leq 0$, i.e., a lender with high ability forms a partnership with an entrepreneur with low collateral.*

Each partnership (c, w) writes a binding *loan contract* that specifies state-contingent transfers $r(0)$ and $r(Q)$ to the investor at $t = 1$. We assume *limited liability* such that in the event of failure, no agent is paid, i.e., $r(0) = 0$. Let $R = r(Q)$ denote the 'loan rate'.

2.2 The timing of events

At date 0, investors and entrepreneurs form partnerships via a one-to-one matching rule. At $t = 1$, each lender makes a take-it-or-leave-it contract offer to a borrower which specifies the loan rate, and decides how much monitoring effort to exert. Finally, at $t = 2$, the true cash flow is realized and the agreed upon payments are made. We solve the model by backward induction.

2.3 Equilibrium

An allocation (λ, v, u) for the lender-borrower market ξ consists of matches (c, w) formed through feasible contracts, and payoff allocations (v, u) such that $v : C \to \mathbb{R}_+$ and $u : W \to \mathbb{R}_+$ are the utilities of the lenders and the borrowers, respectively.

Definition 2 (Equilibrium allocation) An allocation (λ, v, u) is an equilibrium allocation for the investor-entrepreneur economy ξ if the following conditions are satisfied:

(a) **Feasibility:** Given a matching rule λ, for all $w \in W$ the payoff vector $(v(\lambda(w)), u(w))$ must be feasible for the pair $(\lambda(w), w)$, i.e., $u(w) \in [0, u_{max}(\lambda(w), w)]$ where $u_{max}(\lambda(w), w)$ solves $\phi(\lambda(w), w, u) = 0$ and $v(m) \leq \phi(\lambda(w), w, u(w))$.

(b) **Stability:** Given the payoff vectors (v, u), there do not exist pairs (c, w) and $\hat{u} > u(w)$ such that $v(c) < \phi(c, w, \hat{u})$.

(c) **Measure consistency:** For any subinterval $[i_0, i_1] \subseteq I$, let $i_k = G(c_k)$ for $k = 0, 1$, i.e., c_k is the ability of the investor at the i_k-th quantile. Similarly, for any subinterval $[j_0, j_1] \subseteq J$, let $j_h = F(w_h)$ for $h = 0, 1$. If $[c_0, c_1] = \lambda([w_0, w_1])$, then it must be the case that $G(c_1) - G(c_0) = F(w_1) - F(w_0)$.

Part (b) of the above definition asserts that if any investor-entrepreneur pair can make an alternative contractual arrangement which would make both of them strictly better-off, then they would 'block' the current allocation. Thus in an equilibrium allocation, such contracts cannot exist. Part (c), the measure-consistency requirement, is the standard 'demand-supply equality' condition of a continuum economy. Note that the Lebesgue measure of a subinterval $[i_0, i_1]$ of investors is $i_1 - i_0 = G(c_1) - G(c_0)$, and that of a subinterval $[j_0, j_1]$ of entrepreneurs is $j_1 - j_0 = F(w_1) - F(w_0)$. Thus, measure-consistency requires that if $[j_0, j_1]$ is matched to $[i_0, i_1]$, then these two subintervals cannot have different measures.

3 Equilibrium sorting and loan contracts

We proceed as follows. In Section 3.1, we analyze the optimal loan contract for an arbitrary match (c, w). In Section 3.2, we derive that the equilibrium matching is negatively assortative. In Section 3.3 we analyze the behavior of the equilibrium loan rate as a function of borrower collateral.

3.1 Optimal loan contract and the Pareto frontier of an arbitrary match

Any optimal contract for an arbitrary match (c, w) will depend on lender- and borrower-types, but to save on notations, we suppress the argument (c, w) from the contract terms. An optimal loan contract solves the following maximization problem:

$$\max_{\{R,m\}} V(R,m) \equiv mpR - \frac{m^2}{2c} - (1-w), \tag{$\mathscr{M}$}$$

$$\text{subject to } U(R,m) \equiv mp(Q-R) + (1-m)B - w \geq u, \tag{PCB}$$

$$m = \text{argmax}_{\hat{m}} \{mpR - \frac{m^2}{2c} - (1-w)\} = cpR, \tag{ICL}$$

$$0 \leq R \leq Q. \tag{LL}$$

The expected payoffs of the lender and the borrower are respectively denoted by $V(R, m)$ and $U(R, m)$. Constraint (PCB) is the borrower's participation constraint where $u \geq 0$ is his outside option. We assume $B > w + u$ so that the borrower moral hazard problem is not trivial. Constraint (ICL) is the lender's incentive compatibility constraint, and (LL) is the limited liability constraint of the borrower. The following lemma characterizes the optimal contract for an arbitrary partnership (c, w).

Lemma 1 *In an arbitrary partnership (c, w), borrower's participation constraint always binds. Let $m(c, w, u)$ and $R(c, w, u)$ denote respectively the optimal monitoring intensity and loan rate.*

(a) *At the optimal contract there is always over monitoring, i.e., $m(c, w, u) \geq m^{FB}$ where m^{FB} is the first-best level of monitoring. Moreover, the optimal monitoring is monotonically increasing* *in investor ability c, and monotonically decreasing in borrower collateral w and entrepreneur's outside option u;*

(b) *The optimal loan rate is monotonically decreasing in investor ability c, borrower collateral w and entrepreneur's outside option u;*

Consider first the effect of an increase in c. Note that the incentive constraint (ICL) of the lender dictates that his marginal income R must be equal to her marginal cost of monitoring. Because the marginal cost is monotonically decreasing in c, other things being equal, she requires a lower marginal compensation R, and hence, the optimal loan rate decreases with c. By contrast, an increase in ability for a given level of R implies that it is now less costly for the investor to marginally increase monitoring effort, and hence, the optimal monitoring increases. Next, an increase in w implies greater equity participation by the entrepreneur. Because the participation constraint of the entrepreneur binds at the optimum and his expected utility (gross of w) is strictly decreasing in R, the only way to induce the borrower to accept the contract is to lower the loan rate. However, this undermines the incentive for the investor to increase monitoring, and hence, the optimal monitoring intensity decreases. Finally, consider the effect of an increase in the entrepreneur's outside option. In optimal contracting problems under moral hazard and limited liability, it is typical for the investor to face a trade-off between incentive provision and giving up ex ante rent to the borrower. When u increases, the entrepreneur gains greater bargaining power, and hence, the investor is forced to pay him more. Because the participation constraint is binding, this extra payment cannot be given in the form of additional rent; rather, it must be given in the form of stronger incentives. As a consequence, $Q-R$ increases, i.e., R decreases. A decrease in R implies lower monitoring at the optimum.

The object of our interest will be the Pareto frontier associated with an arbitrary partnership (c, w), denoted by $\phi(c, w, u)$, which is the maximum value function of the above maximization problem. In the following lemma we state an useful property of the frontier function.

Lemma 2 *The Pareto frontier $\phi(c, w, u)$ of an arbitrary match (c, w) satisfies the following single-crossing condition:*

$$\frac{\partial}{\partial c}\left[-\frac{\phi_2(c, w, u)}{\phi_3(c, w, u)}\right] < 0. \tag{SC}$$

The above single-crossing condition is depicted in Figure 1.

The slope of the indifference curve of any investor in the (w, u) space is given by $-\phi_2/\phi_3$. Condition (SC) asserts that the slope of the indifference curve of any lender is decreasing in lender-type, and hence, the indifference curves of any two investors with different types can cross only once.

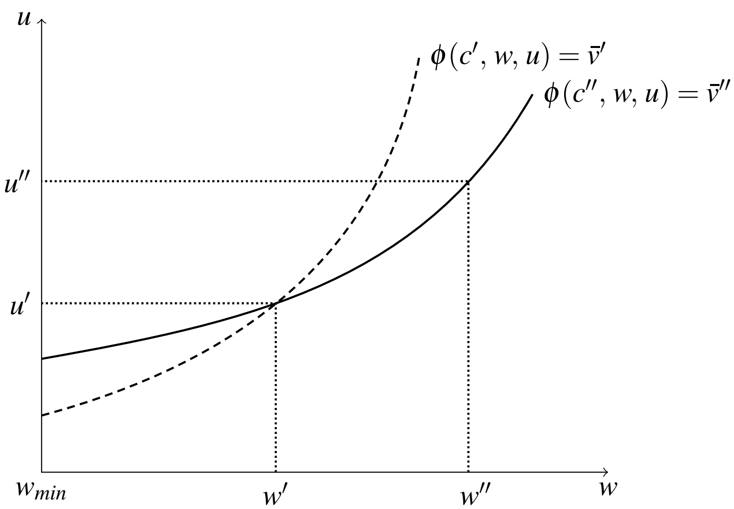

Figure 1: *Single-crossing condition: the indifference curve of lender-type c'' is flatter everywhere than that of c' where $c'' > c'$.*

3.2 Equilibrium payoffs and matching

Definition 2-(b) implies that, given $u(w)$, each type c lender solves the following maximization problem

$$\lambda^{-1}(c) = \text{argmax}_w\, \phi(c, w, u(w)). \tag{$\mathscr{M}$}$$

The first-order condition of the above maximization problem is given by the following ordinary differential equation:

$$u'(w) = -\frac{\phi_2(c, w, u(w))}{\phi_3(c, w, u(w))} \equiv \psi(c, w, u(w)) \quad \text{for } c = \lambda(w). \tag{U}$$

It follows from the Envelope theorem that

$$v'(c) = \phi_1(c, w, u(w)) \qquad \text{for } c = \lambda(w). \tag{V}$$

It is easy to show that $\phi_1, \phi_2 > 0$ and $\phi_3 < 0$, and hence, $u'(w) > 0$ and $v'(c) > 0$. Let $u_0 \geq 0$ be the reservation utility of all entrepreneurs irrespective of the level of collateral, which represents the utility obtained by any entrepreneur if his project is not financed by any investor. Note that the notion of outside option differs from that of reservation utility, which is an exogenous object. In an equilibrium allocation, we must have $u(w) \geq u_0$ for all $w \in W$. This is the "individual rationality" condition that has been omitted in the maximization problem ($\mathscr{M}$). Because $u(w)$ is strictly increasing in w, u_0 must also be the outside option of the entrepreneur with the minimum collateral $w = w_{min}$, i.e., $u(w_{min}) = u_0$. Therefore, the equilibrium payoff function $u(w)$, which is the solution to the differential equation (U), is given by the following:

$$u(w) = u_0 + \int_{w_{min}}^{w} \psi(\lambda(x), x, u(x))dx. \tag{U'}$$

The above condition implies that $u(w) - u_0$ is the area below the curve $\psi(\lambda(w), w, u(w))$ between w_{min} and w, and hence, $u(w) > u_0$ for all $w > 0$, i.e., all borrower types except the one with the lowest collateral earn type-specific rents. Next, we show that in any equilibrium allocation the matching is negative assortative which is an immediate consequence of the single-crossing condition (SC).

Proposition 1 *In any equilibrium allocation* (λ, v, u)*, we have* $\lambda'(w) \leq 0$*, i.e., more efficient lenders (monitors) invest in firms with lower collateral following a negative assortative matching pattern.*

Condition (SC) implies that if a lender with higher type c'' is indifferent between the firm type-payoff combinations (w', u') and (w'', u'') with $w'' > w'$ and $u'' > u'$, then a lower type c' prefers to pay more [than u''] to borrower of type w'', i.e., for any $c'' > c'$,

$$\phi(c'', w', u') = \phi(c'', w'', u'') \implies \phi(c', w'', u'') \geq \phi(c', w', u'). \tag{GDD}$$

The above is the *generalized decreasing difference* (GDD) condition in Legros and Newman (2007) which implies that lender and borrower types are substitutes, and is a necessary and sufficient condition for NAM. Condition (GDD) is weaker than the usual substitutability implied by the *decreasing differences* or *submodularity* of the Pareto frontier.[6]

The phenomenon of endogenous lender-borrower matching is ubiquitous in financial markets (e.g. Chen, 2013). However, empirical evidence of negative sorting in such markets has been scarce until recently. Schwert (2018) finds evidence of NAM in a syndicated corporate loan market – bank-dependent borrowers (low equity capital or collateral) tend to secure funding from well-capitalized banks who, according to the 'equity monitoring hypothesis' (e.g. Hölmstrom and Tirole, 1997; Mehran and Thakor, 2011), have stronger incentives to monitor borrowers. Therefore, Schwert's (2018) finding supports our theoretical prediction that lenders with stronger incentives to monitor sort themselves into borrowers with lower collateral.

3.3 Equilibrium loan rates

Recent empirical literature on principal-agent matching (e.g. Ackerberg and Botticini, 2002) claims that optimal incentive contracts under endogenous matching can be very different from those predicted by the standard agency theory that treats a principal-agent partnership in isolation. To understand this point, suppose that there are two types of entrepreneurs (those with high and low collateral, i.e., $w_2 > w_1$),

[6] The maximization problem ($\mathscr{M}$) is similar to incentive compatibility in the 'optimal screening problems' (e.g. Maskin and Riley, 1984), where the first-order condition (U) is a kind of *local downward incentive constraint*. Moreover, equation (U′) is similar to the *informational rent* of an agent in an optimal screening problem, which is monotonically increasing in w.

each with measure 0.5, and two types of investors (those with high and low monitoring ability, i.e., $c_2 > c_1$), each with measure 0.5. Because the equilibrium matching is negative assortative, the two types of partnerships that will be formed are (c_1, w_2) and (c_2, w_1). Standard agency models would predict that a higher loan rate must be associated with lower collateral, i.e., the loan rate associated with the match (c_2, w_1) must be higher. But this is purely a partial equilibrium phenomenon where the difference in loan rate is implied only by the difference in collateral values. Lemma 1-(b) also predicts that the loan rate for (c_2, w_1) should be lower because this match involves greater monitoring efficiency. Thus, the (market) equilibrium loan rate for (c_2, w_1) may be higher or lower than that associated with (c_1, w_2) depending on which of the two aforementioned countervailing forces is stronger. Therefore, the outcome of an assignment model offers predictions about the equilibrium loan rate with respect to borrower collateral that may be exactly the opposite of what would have been predicted by standard agency theory. This simple example regarding the general non-monotonicity of the loan rate with respect to collateral motivates us to analyze the behavior of the equilibrium loan rate.

An increase in the borrower collateral in our general equilibrium framework affects the equilibrium loan rate through three channels (one direct and two indirect) as followed from Lemma 1-(b). First, a direct channel, i.e., greater collateral implies a lower optimal loan rate to be paid to the lender. Second, collateral affects the loan rate through the equilibrium matching function, i.e., a higher value of collateral in a given partnership implies a lower monitoring ability because of NAM, and hence, must be associated with a higher loan rate as less efficient monitors require greater marginal compensation. Third, a change in collateral value alters a borrower's bargaining power via the equilibrium utility function $u(w)$, and a greater bargaining power caused by higher collateral implies lower loan rates. Clearly, the three aforementioned effects do not point in the same direction, yielding a potential non-monotonic behavior of the equilibrium loan rate with respect to borrower collateral. To see this formally, let the equilibrium loan rate function be given by:

$$R(w) \equiv R(\lambda(w), w, u(w)).$$

Differentiating the above with respect to w, we obtain the following:

$$R'(w) = \underbrace{\frac{\partial R}{\partial c} \cdot \lambda'(w)}_{(+)} + \underbrace{\frac{\partial R}{\partial w}}_{(-)} + \underbrace{\frac{\partial R}{\partial u} \cdot u'(w)}_{(-)}. \tag{1}$$

The first term is positive because both $\partial R/\partial c < 0$ and $\lambda'(w) \leq 0$. The second and third terms, however, are negative because $\partial R/\partial w = \partial R/\partial u < 0$ and $u'(w) > 0$. Thus, the relationship between the equilibrium loan rate and collateral is in general non-monotonic, and the monotonicity of the equilibrium loan rate with respect to w depends on which of the two countervailing effects dominates.

Proposition 2 *The equilibrium loan rate $R(w)$ is in general non-monotonic with respect to borrower* *collateral. If*

$$\frac{f(w)}{g(c)} > (<)\, \chi(c,w,u) \;\; \textit{for all } (c,w,u), \tag{$\star$}$$

where

$$\chi(c,w,u) \equiv \frac{c^2 \left[4(B-w-u) + c(pQ-B)\left(c(pQ-B) - \sqrt{c^2(pQ-B)^2 + 4c(B-w-u)}\right)\right]}{2(B-w-u)^2},$$

then $R(w)$ is monotonically increasing (decreasing) in w.

Condition ($\star$), which is derived in the Appendix D, is not particularly intuitive. Note that (1) implies that the absolute value of $\lambda'(w)$, the slope of the equilibrium matching function must be high (low) enough so that the equilibrium loan rate is monotonically increasing (decreasing) in w. Now, suppose that a type w entrepreneur is matched with a type c investor. An immediate consequence of Definition 2-(b) and NAM is that $G(c) = 1 - F(w)$, i.e., $c = G^{-1}(1 - F(w)) \equiv \lambda(w)$. Therefore,

$$\left|\lambda'(w)\right| = \frac{f(w)}{g(\lambda(w))}.$$

Thus, if the left-hand-side of ($\star$) is sufficiently high, then the equilibrium matching function is steep enough, and consequently, $R'(w) > 0$. Also, the sufficient condition for monotonicity of the loan rate with respect to borrower collateral is a stringent condition which requires that ($\star$) holds for all possible matches (c,w) and all values of borrower outside option u, and hence, is difficult to verify. One possible intuitive explanation [among many] is the following. Consider a scenario in which the borrowers are very heterogenous relative to the lenders. Thus, the matching function $\lambda(w)$ is relatively flat. Then, the differences in the equilibrium loan rate will be largely explained by the differences in borrower collateral across matches, and hence, condition ($\star$) would imply a negative association between loan rate and collateral. By contrast, if lenders are very heterogeneous relative to the borrowers, the differences in monitoring ability will be much more important than those in collateral values in explaining the differences in the loan rate, and consequently, $R(w)$ will be monotonically increasing in w. In Figure 3, we present numerical examples with permissible parameter values, and specific type distributions, in which the equilibrium loan rate is non-monotonic in borrower collateral.

Berger et al. (2016) find empirical evidence of a non-monotonic relation between loan rate and collateral. They argue that collateral may have different desirable economic characteristics, such as liquidity and outside ownership status, each of which may influence loan risk in a different way; hence, the empirical relation between the loan rate (or equivalently, loan risk premium) and collateral may not be monotonic. Our alternative explanation for the non-monotonicity of loan rate is based on two-sided heterogeneity and endogenous matching in which the equilibrium matching pattern and endogenous payoffs play crucial roles.

4 Effect of cross-sectional variations in the type distribution

The property that any equilibrium allocation of the lender-borrower market exhibits NAM depends only on the single-crossing condition (GDD) of the Pareto frontier, and not on the distributions of types, $G(c)$ and $F(w)$. However, the measure consistency condition together with the cumulative distributions of types pin down the equilibrium matching function. Therefore, although the pattern of equilibrium matching is distribution-free, the shape of the matching function is not. Such dependence of the equilibrium assignment on the type distributions allows us to conduct meaningful comparative statics exercises, namely, the effects of changes in the type distributions on the equilibrium lender-borrower contracts.

4.1 Effect on equilibrium matching and borrower utility

We first analyze the effects of changes in type distributions on the equilibrium matching function $\lambda(w)$ and the borrower payoff function $u(w)$. Because only the relative density $f(w)/g(c)$ is relevant to the shape of the matching function, henceforth, we assume without loss of generality that c is uniformly distributed on C. In this case, $g(c)$ is constant for all c, and the equilibrium matching function is a (scaled) mirror image of $F(w)$. Therefore, a change in $F(w)$ would affect both $\lambda(w)$, and $u(w)$ via shifts in the matching function. Consider two lender-borrower economies $\xi^1 = (F_1, G)$ and $\xi^2 = (F_2, G)$, which differ only in the distributions of collateral, and assume without loss of generality that $W = [w_{min}, w_{max}] = [0, 1]$. Recall that

Definition 3 (Second-order stochastic dominance) *The distribution F_1 is said to second-order stochastically dominate F_2, i.e., $F_1 \succ_{SOSD} F_2$ if there is a unique $w^* \in (0, 1)$ such that*

$$F_1(w) < (>)\, F_2(w) \;\; \textit{for all} \;\; w < (>)\, w^*,$$
$$\textit{and} \;\; \int_0^1 [F_1(x) - F_2(x)]dx = 0.$$

Note that $F_1 \succ_{SOSD} F_2$ implies that the distribution of collateral in ξ^2 is more unequal than that in ξ^1.[7] In the following proposition, we state our main comparative statics result.

Proposition 3 *Consider two distinct lender-borrower economies ξ^1 and ξ^2 that differ only in the distributions F_1 and F_2 of collateral. Furthermore, let $\lambda_k(w)$ be the equilibrium matching function and $u_k(w)$ be the equilibrium payoff of each type w entrepreneur in economy ξ^k for $k = 1, 2$. If $F_1 \succ_{SOSD} F_2$, then $\lambda_1(w) > (<)\, \lambda_2(w)$ for $w < (>) w^*$, and either* (i) *$u_1(w) < u_2(w)$ for all $w \in (0, 1]$ or* (ii) *there is a unique $\hat{w} \in (w^*, 1)$ such that $u_1(w) < (>)\, u_2(w)$ for $w < (>)\, \hat{w}$.*

[7] This is a restrictive definition of second-order stochastic dominance, but is the most popular one. Definition 3 asserts that the distribution functions $F_1(w)$ and $F_2(w)$ cross one another only once, and they have the same expectation. Second-order stochastic dominance may imply multiple crossings which we do not consider as the results of the analysis would not be very tractable.

The nature of the shift in the equilibrium matching function resulting from a change in the distribution of collateral is trivial. Because the equilibrium matching function in any lender-borrower economy is a mirror image of the cumulative distribution function of w, it must be the case that $\lambda_h(w) > \lambda_k(w)$ if and only if $F_h(w) < F_k(w)$ for $h, k = 1, 2$ and $h \neq k$. Therefore, if for any given $w \in W$, we have $F_1(w) < (>) F_2(w)$, then, any entrepreneur of type w must be matched with an investor with lower (higher) monitoring ability in the equilibrium of ξ^2.

The nature of the shift in the equilibrium (borrower) utility function $u(w)$ following a change in the type distribution function $F(w)$ is less trivial. First, let us discuss the channel through which a change in the distribution of collateral affects the equilibrium payoff of an entrepreneur. Note that $\lambda_1(w^0) > \lambda_2(w^0)$ for a given $w^0 \in W$ implies that $\psi(\lambda_1(w^0), w^0, u(w^0)) < \psi(\lambda_2(w^0), w^0, u(w^0))$ because the single-crossing condition (SC) holds for all (c, w). Because $u(w) - u_0$ is the area under the curve $\psi(\lambda(w), w, u(w))$ [cf. (U$'$)], we have $u_2(w^0) > u_1(w^0)$. In other words, a change in the distribution of collateral shifts bargaining power from one side of the market to the other. With a change in the distribution of collateral in the sense of second-order stochastic dominance, the shift in bargaining power from entrepreneurs to investors is asymmetric because under such a change, the $\psi(\lambda(w), w, u(w))$ curve rotates down around some w. Therefore, for low values of collateral, the bargaining power of associated lenders increases, whereas for high values of collateral, the borrowers gain increased bargaining power. The results of Proposition 3 are obtained depending on which of the two countervailing forces dominates.

4.2 Implications for the equilibrium loan rate and monitoring: numerical results

As the equilibrium loan rate and monitoring are expressed as a function of collateral, any change in $F(w)$ would also change these equilibrium variables. In our static equilibrium model, such an exercise should be viewed as the effect of cross-sectional variations in the investor-entrepreneur market. To analyze the effect of a change in the distribution of collateral on $R(w)$, the equilibrium loan rate, one must first solve the ordinary differential equation (ODE) in (U). When we substitute for $\phi_2(m, w, u(w))$ and $\phi_3(m, w, u(w))$, the ODE reduces to the following:

$$u'(w) = \psi(\lambda(w), w, u(w)) = 1 - \frac{pQ - B}{pR(\lambda(w), w, u(w))}.$$

Under imperfectly transferable surplus, the Pareto frontier is a non-linear function of $u(w)$, and hence, the above ODE does not have an analytical solution. Moreover, it is difficult to determine the direction of shifts in the equilibrium loan rate following a change in $F(w)$ for the following reason. Let $R_k(w) \equiv R(\lambda_k(w), w, u_k(w))$ be the equilibrium loan rate associated with a common collateral value w in economy

ξ^k for $k = 1, 2$. Then,

$$\Delta R(w) \equiv R_2(w) - R_1(w) \approx \underbrace{\frac{\partial R}{\partial c}\left[\lambda_2(w) - \lambda_1(w)\right]}_{\text{matching effect}} + \underbrace{\frac{\partial R}{\partial u}\left[u_2(w) - u_1(w)\right]}_{\text{utility effect}}. \tag{2}$$

In the above expression, a change in the equilibrium loan rate is decomposed into two effects – namely, a *matching effect* and an *utility effect* which in general point in opposite directions. To see this, consider the simplest case in Proposition 3 that $F_1 \succ_{SOSD} F_2$ implies $\lambda_1(w) > (<) \lambda_2(w)$ for $w < (>) w^*$, and $u_1(w) < u_2(w)$ for all $w \in (0, 1]$. In this case, both terms of (2) are negative for $w > w^*$, and hence, $R_2(w) < R_1(w)$ for all $w > w^*$. However, for $w < w^*$, the first term of (2) is positive because $\partial R/\partial c < 0$ and $\lambda_2(w) < \lambda_1(w)$, but the second term is always negative because $\partial R/\partial u < 0$ and $u_2(w) > u_1(w)$. Therefore, the effect of second-order stochastic dominance on the equilibrium loan rate is ambiguous for values of $w < w^*$. Similar argument applies as far as the change in the equilibrium monitoring intensity $m(w)$ following a change in $F(w)$ is concerned.

We therefore resort to a numerical simulation of the model to examine the shifts in the equilibrium loan rate $R(w)$ and monitoring $m(w)$ when $F_1 \succ_{SOSD} F_2$. For this purpose, we assume that $p = 0.5$, $Q = 4.3$, $B = 2.1$, and $u_0 = 0.1$. Further, monitoring ability c is assumed to be uniformly distributed on $[0.05, 0.4]$, and borrower collateral w is assumed to follow a beta distribution with parameters α and β, i.e.,

$$F(w) = \frac{\int_0^w x^{\alpha-1}(1-x)^{\beta-1}dx}{\int_0^1 w^{\alpha-1}(1-w)^{\beta-1}dw}, \quad \alpha, \beta > 0.$$

There are two principal reasons for choosing a beta distribution. First, one must define a bounded support for collateral, w, which is consistent with the model's assumptions. Second, as we have seen from the theoretical analysis, the heterogeneity in c and w is crucial to determine the relative importance of the matching and utility effects on the equilibrium loan rate; thus, a beta distribution is sufficiently flexible to consider alternative specifications for the relative heterogeneity between investors and entrepreneurs. As we have discussed earlier, because two comparable markets differ only in the corresponding distributions of collateral, there is no loss of generality in assuming a uniform distribution of monitoring ability.

To give an empirical content to the analysis, we use the data constructed by Angelini and Generale (2008) for a sample of Italian firms to match the beta distribution parameters α and β.[8] The dataset includes four surveys run in 1992, 1995, 1998 and 2001 by an Italian credit institution. Each survey includes information about several characteristics of the firm including the value of its assets. The surveys are representative of Italian manufacturing firms with more than ten employees. One major advantage of this dataset is that firms with less than 50 employees are well represented in the surveys. Because we have assumed that the entrepreneurs can pledge his entire initial wealth as collateral, we take the natural logarithm of total asset value (i.e., current plus net fixed assets) as our proxy for borrower collateral. For

[8]The dataset is available at https://www.aeaweb.org/articles?id=10.1257/aer.98.1.426.

illustrative purposes, we select only the 1992 and 1995 surveys. After cleaning up the data, we are left with 1,568 and 2,438 observations, respectively. The corresponding cumulative distribution functions of total asset value (in logs) are shown in Figure 2. A casual inspection of Figure 2 suggests that the asset distribution of 1995 second-order stochastically dominates that of 1992. This is statistically confirmed by applying the dominance test proposed by Linton, Maasoumi, and Whang (2005) to both samples.

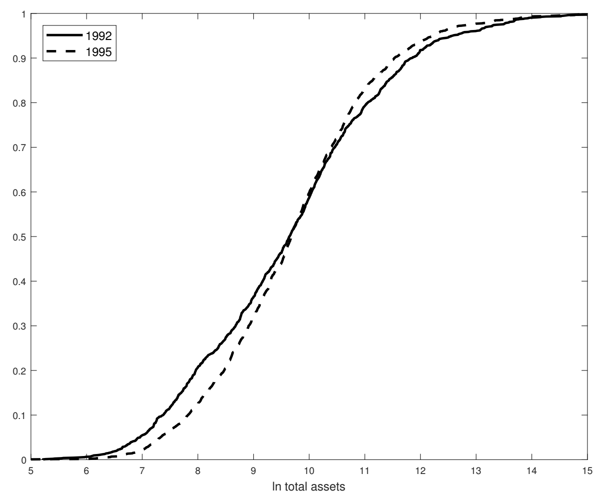

Figure 2: *The empirical cumulative distribution of borrower collateral of 1995 second-order stochastically dominates that of 1992.*

The data in Figure 2 can be conveniently represented by a beta distribution. This requires normalizing the total asset values on the support $[0, 1]$. For example, the collateral value 9.9 in the data corresponds to 0.57 under the normalized distribution. The problem with this procedure is that we do not know a priori the values of the two parameters α and β of the distribution of collateral. We proceed as follows. First, we estimate the mean and variance of each sample. Next, we use these two moments to calibrate the two (unknown) parameters of each distribution, so that the first two moments of the "artificial" distribution replicate the moments observed in the data. The estimated parameters of the beta distribution corresponding to 1992 and 1995 are given by $\alpha_{1992} = 7.89$, $\beta_{1992} = 6.81$ and $\alpha_{1995} = 10.93$, $\beta_{1995} = 9.24$.

The dataset does not have information on loan contracts. So, we construct the functions $R(w)$ and $m(w)$ using the aforementioned parameter values. The simulated profiles of the equilibrium loan rate and monitoring are presented in Figures 3 and 4, respectively. It is worth mentioning that the support of total asset value has been shrunk for illustrative purposes (there are no significant differences in loan rates and monitoring intensities between the two samples outside the interval shown). The equilibrium loan rate function is clearly non-monotonic in borrower collateral for each sample. The equilibrium loan rate in the 1992 sample is higher for values of collateral between 0.3 and 0.57. On the other hand, the equilibrium monitoring intensity in the 1992 sample is lower than that of 1995 for collateral values between 0.33 and 0.57. Thus, from the numerical exercise we conclude that

Result 1 (Change in the inequality of collateral) *Consider two economies, $\xi^1 = (F_1, G)$ and $\xi^2 = (F_2, G)$, where $F_1 \succ_{SOSD} F_2$, i.e., the distribution of collateral in ξ^2 is more unequal than that in ξ^1. Then, there is a unique $\bar{w}_R$ such that an entrepreneur with $w < (>)\, \bar{w}_R$ pays a higher (lower) loan rate in the equilibrium of ξ^2. Moreover, there is a unique $\bar{w}_m$ such that an entrepreneur with $w < (>)\, \bar{w}_m$ suffers lower (higher) monitoring in the equilibrium of ξ^2.*

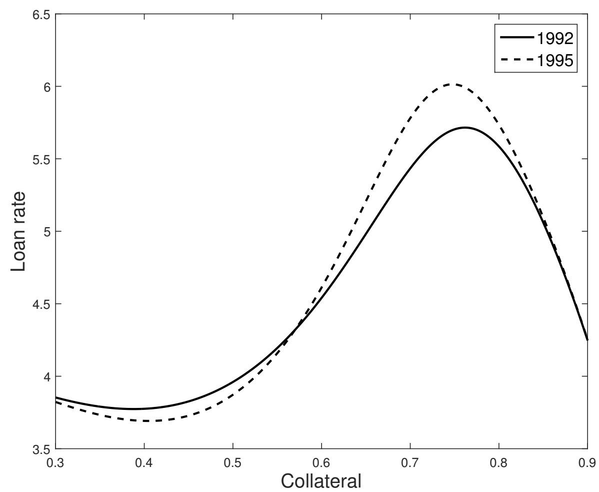

Figure 3: *The simulated equilibrium loan rate functions corresponding to years 1992 and 1995.*

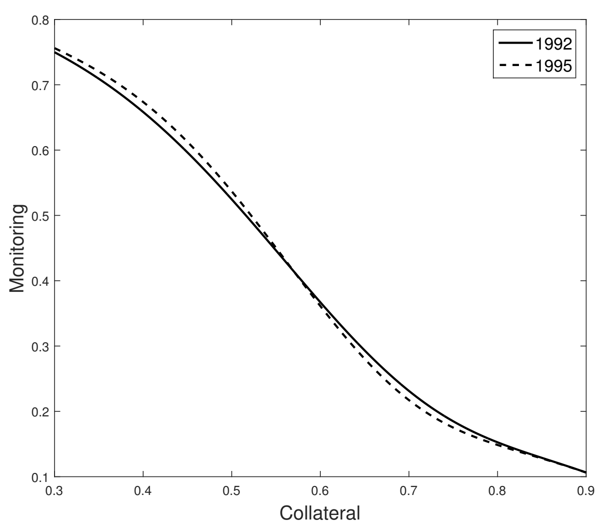

Figure 4: *The simulated equilibrium monitoring functions corresponding to years 1992 and 1995.*

The intuition follows from the cumulative distribution functions in Figure 2. Borrowers with collateral values of 9.9 (0.57 in the normalized distribution) or less are more abundant in the 1992 sample than in the 1995 sample. As a result, their bargaining power is lower and thus pay higher loan rates compared to those in the 1995 sample. By contrast, entrepreneurs with collateral values of 9.9 or more are relatively less abundant in the 1992 sample. For such collateral levels, the bargaining power of entrepreneurs is greater in the 1992 sample than the 1995 sample, which translates into a lower loan rate in the 1992 sample. Thus, the overall effect of such change in the distribution is determined by the resulting change in the bargaining power of entrepreneurs. In a more unequal economy, ξ^2, entrepreneurs with high collateral have greater bargaining power because the competition for better entrepreneurs is more fierce compared with the competition in ξ^1. Conversely, low-collateral borrowers have lower bargaining power in economy ξ^2. Therefore, high-collateral entrepreneurs pay lower loan rates in economy ξ^2, whereas low-collateral borrowers face an increased cost of external financing in ξ^2. On the other hand, high-collateral entrepreneurs suffers from higher monitoring in economy ξ^2.

5 Concluding remarks

Compared with contracts for an isolated investor-entrepreneur pair, incentive contracts may be quite different in a market with many heterogeneous investors and entrepreneurs. In the equilibrium of a market, individual contracts are influenced by the two-sided heterogeneity via endogenous investor-entrepreneur matching. In this paper, we have developed a simple two-sided matching model of incentive contracting between lenders and borrowers. Entrepreneurs who differ in collateral values and investors who differ in monitoring ability are matched into pairs in order to accomplish projects of fixed size. In the equilibrium of the market, both the matching and payoffs that accrue to each individual are determined endogenously. We show that the equilibrium lender-borrower matching is negatively assortative, and there is a general non-monotonic association between the loan rate and collateral. Further, we show how a change in the distribution of collateral may affect asymmetrically the cost of external funding and monitoring.

Although our stylized model is built on a number of simplifying assumptions, our conclusions are somewhat general, and can be extended to credit relationships other than those analyzed in the present paper. Under double-sided moral hazard, in each investor-entrepreneur partnership, what is crucial is the identification of lenders and borrowers who are "easier to incentivize". When investors perform the role of monitors, in equilibrium, investors with stronger incentives to monitor form partnerships with borrowers with greater need of outside equity following a negative assortative matching pattern. Thus, any empirical analysis that finds direct or indirect evidence of such negative sorting (e.g. Schwert, 2018) by using a set of appropriate measures is consistent with our theoretical finding.

Our model is an extension of Shapley and Shubik (1971) to an environment where matches are pervaded by the underlying incentive problems, which give rise to a non-linear Pareto frontier. However, the agency problem is not the only fundamental that induces imperfect transferability. If at least one of the two contracting parties is risk averse, then the associated frontier is concave even without the incentive problem. In such markets, as argued by Legros and Newman (2007), partnerships are formed because of pure risk-sharing motives, whereas in our model, each partnership (which is subject to limited liability) implies an optimal trade-off between the provision of incentives and rent extraction. Therefore, an extension of the current paper to an environment with risk-averse individuals may shed light on the implications of risk-averse investors in the corporate loan markets.

A more ambitious model would consider many-to-many matching among investors and entrepreneurs. When a lender is allowed to invest in more than one firm, additional complications arise because the monitoring cost function is in general not additively separable. Thus, non-zero interaction terms induce externalities across matches. On the other hand, allowing an entrepreneur to borrow from multiple sources may imply the inability of lenders to write binding exclusive contracts. Non-exclusivity may also lead to an externality across matches. Second, an important assumption in the paper is that the relationship between an investor and an entrepreneur lasts only for one period. Such relationships could also involve dynamic considerations, which in turn imply some degree of relaxation of the limited liability constraints, and the conclusions of the current paper could thus change. In a dynamic model, when there are possibilities of wealth accumulation, the income distributions of an economy are generally endogenous. The literature on two-sided matching (e.g. Shapley and Shubik, 1971) has largely been silent on the context of dynamic bilateral relationships. In this context, the papers by Mookherjee and Ray (2002) is worth mentioning. Mookherjee and Ray (2002) analyze a dynamic model of equilibrium short-period credit contracts when lenders and borrowers are randomly matched, and the bargaining power is exogenously given. When lenders have all the bargaining power, less wealthy borrowers have no incentive to save, and poverty traps emerge. Conversely, if borrowers have all the bargaining power, income inequality is reduced as a result of the strong incentives for savings. One significant difference between our model and that of Mookherjee and Ray (2002) is that bargaining power in the current model is distributed endogenously among the principals and agents because the outside option of each individual is endogenous.

Appendix

A Proof of Lemma 1

We analyze the optimal loan contract for an arbitrary partnership (c, w), which solves the program $(\mathscr{M})$. We ignore the limited liability constraint (LL) for the time being. Substituting $m = cpR$ into the objective function and the participation constraint, the maximization problem $(\mathscr{M})$ reduces to:

$$\max_m V(m) \equiv \frac{m^2}{2c} - (1-w), \tag{$\mathscr{M}'$}$$

$$\text{subject to } U(m) \equiv m(pQ-B) - \frac{m^2}{c} + B - w \geq u. \tag{PCB$'$}$$

The Lagrangean is given by:

$$\mathscr{L} = \frac{m^2}{2c} - (1-w) + \mu\left(m(pQ-B) - \frac{m^2}{c} + B - w - u\right),$$

where μ is the associated multiplier. The first-order condition with respect to m yields

$$\mu = \frac{m}{2m - c(pQ-B)}.$$

Clearly, $2m \geq c(pQ-B)$ for μ to be non-negative, and hence, $m > 0$ and $\mu > 0$. Thus, (PCB′) binds at the optimum, which determines the optimal monitoring which is given by:

$$m(c,w,u) = \frac{1}{2}\left(c(pQ-B) + \sqrt{c^2(pQ-B)^2 + 4c(B-w-u)}\right). \tag{3}$$

We ignore the smaller root of m because m must be greater than $c(pQ-B)/2$. Note that the first-best monitoring is given by:

$$m^{FB} = \operatorname*{argmax}_m \left\{mpQ + (1-m)B - \frac{m^2}{2c} - 1\right\} = c(pQ-B) \tag{4}$$

Under the assumption that $B \geq w + u$, it follows from (3) that $m(c,w,u) \geq m^{FB}$, i.e., at the optimal second-best contract there is over monitoring, which also implies that $\mu \leq 1$. The optimal loan rate is determined by substituting $m = cpR$ into (3), which is given by:

$$R(c,w,u) = \frac{m(c,w,u)}{cp} = \frac{1}{2cp}\left(c(pQ-B) + \sqrt{c^2(pQ-B)^2 + 4c(B-w-u)}\right). \tag{5}$$

Now differentiating (3) with respect to c, w and u, respectively we obtain

$$\frac{\partial m}{\partial c} = \frac{m\mu}{c} > 0, \quad \frac{\partial m}{\partial w} = \frac{\partial m}{\partial u} = -\frac{c\mu}{m} < 0.$$

The above inequalities prove Part (a). Finally, differentiating (5) we get

$$\frac{\partial R}{\partial c} = -\frac{m(1-\mu)}{c^2 p} < 0, \quad \frac{\partial R}{\partial w} = \frac{\partial R}{\partial u} = -\frac{\mu}{pm} < 0.$$

This completes the proof of the lemma.

B Proof of Lemma 2

The Pareto frontier, which is the value function of the program ($\mathscr{M}'$) with (PCB$'$) binding is given by:

$$\phi(c,w,u) = \frac{1}{2c}\cdot(m(c,w,u))^2 - (1-w). \tag{PF}$$

It follows from the Envelope theorem that

$$\phi_2(c,w,u) = \frac{\partial \mathscr{L}}{\partial w} = 1-\mu > 0, \quad \phi_3(c,w,u) = \frac{\partial \mathscr{L}}{\partial u} = -\mu < 0.$$

Note that

$$\frac{\partial \mu}{\partial c} = \frac{\partial}{\partial c}\left\{\frac{m}{2m - c(pQ-B)}\right\} = \frac{m(pQ-B)(1-\mu)}{[2m-c(pQ-B)]^2} > 0. \tag{6}$$

Thus, condition (SC) follows from (6).

C Proof of Proposition 1

The proof of NAM in any equilibrium allocation directly follows from Legros and Newman (2007). Take $c'' > c'$ and $w'' > w'$, and write $u' = u(w')$ and $u'' = u(w'')$, the corresponding equilibrium utilities. Suppose that condition (GDD) holds for all matches but for the aforementioned two lender-borrower pairs, in an equilibrium allocation, NAM does not hold, i.e., $c' = \lambda(w')$ and $c'' = \lambda(w'')$. Assume without loss of generality that $\phi(c'', w', u') = \phi(c'', w'', u'')$. Because (U) holds, i.e., $u'(w) > 0$ for all w, we have $u'' > u'$. Thus, it follows from (GDD) that $\phi(c', w'', u'') \geq \phi(c', w', u')$ which contradicts the fact that

$$w' = \underset{w}{\operatorname{argmax}}\, \phi(c', w, u(w)).$$

This completes the proof of the proposition.

D Proof of Proposition 2

Note that (U) implies

$$u'(w) = -\frac{\phi_2(c,w,u(w))}{\phi_3(c,w,u(w))} = \frac{1-\mu}{\mu}.$$

On the other hand, $\lambda'(w) = -f(w)/g(\lambda(w))$. Using the above facts, and the expressions for $\partial R/\partial c$, $\partial R/\partial w$, $\partial R/\partial u$ and μ, we get

$$R'(w) = \frac{m(\lambda(w),w,u(w))}{\lambda(w)^2 p}\cdot\frac{m(\lambda(w),w,u(w)) - \lambda(w)(pQ-B)}{2m(\lambda(w),w,u(w)) - \lambda(w)(pQ-B)}\cdot\frac{f(w)}{g(\lambda(w))} - \frac{1}{pm(\lambda(w),w,u(w))}.$$

The above is an expression in terms of the endogenous variables $\lambda(w)$ and $u(w)$. A sufficient condition [in terms of the fundamentals] for $R'(w)$ to be positive is, for all (c, w, u),

$$\frac{m(c,w,u)}{c^2} \cdot \frac{m(c,w,u) - c(pQ-B)}{2m(c,w,u) - c(pQ-B)} \cdot \frac{f(w)}{g(c)} - \frac{1}{m(c,w,u)} > 0 \quad \Longleftrightarrow \quad \frac{f(w)}{g(c)} > \chi(c,w,u).$$

The expression for $m(c, w, u)$ is given in (3).

E Proof of Proposition 3

The proof can be easily adapted from (Määttänen and Terviö, 2014, Proposition 4) who analyze a similar result for one-sided matching market. Note that

$$F_1(w) < (>) F_2(w) \text{ for } w < (>) w^* \implies \lambda_1(w) > (<) \lambda_2(w) \text{ for } w < (>) w^*.$$

Moreover, in the equilibria of both markets, $u_1(0) = u_2(0) = u_0$. Therefore,

$$\Delta u(w) \equiv u_1(w) - u_2(w) = \int_0^w [\psi(\lambda_1(w), w, u(w)) - \psi(\lambda_2(w), w, u(w))] dw.$$

As $\psi(m, w, u(w))$ is strictly decreasing in m, the integrand of the above expression, which is the slope of $\Delta u(w)$, is strictly negative (positive) for $w < (>) w^*$. At $w = w^*$, the above definite integral is strictly negative because $\Delta u(0) = 0$, and it is strictly decreasing on $[0, w^*)$. Because $\Delta u(w)$ is strictly increasing on $(w^*, 1]$, there are two possibilities: (i) $\Delta u(w)$ does not intersect the horizontal axis, and hence, it is strictly negative for all $w \in [0, 1]$. Otherwise, (ii) $\Delta u(w)$ intersects the horizontal axis at some point $\hat{w} \in (w^*, 1]$, which is unique because $\Delta u(w)$ is strictly increasing on $(w^*, 1]$.

References

Ackerberg, D. and M. Botticini (2002), "Endogenous Matching and the Empirical Determinants of Contract Form." *Journal of Political Economy*, 110, 564–591.

Agarwal, S. and R. Hauswald (2010), "Distance and Private Information in Lending." *The Review of Financial Studies*, 23, 2757–2788.

Almazan, A. (2002), "A Model of Competition in Banking: Bank Capital vs Expertise." *Journal of Financial Intermediation*, 11, 87–121.

Almazan, A., J. Hartzell, and L. Starks (2005), "Active Institutional Shareholders and Costs of Monitoring: Evidence from Executive Compensation." *Financial Management*, 34, 5–34.

Alonso-Paulí, E. and D. Pérez-Castrillo (2012), "Codes of Best Practice in Competitive Markets for Managers." *Economic Theory*, 49, 113–141.

Angelini, P. and A. Generale (2008), "On the Evolution of Firm Size Distributions." *The American Economic Review*, 98, 426–438.

Berger, A., S. Frame, and V. Ioannidou (2016), "Reexamining the Empirical Relation between Loan Risk and Collateral: The Roles of Collateral Liquidity and Types." *Journal of Financial Intermediation*, 26, 28–46.

Berger, A., N. Miller, M. Petersen, R. Rajan, and J. Stein (2005), "Does Function Follow Organizational Form? Evidence from the Lending Practices of Large and Small Banks." *Journal of Financial Economics*, 76, 237–269.

Berger, A. and G. Udell (1990), "Collateral, Loan Quality, and Bank Risk." *Journal of Monetary Economics*, 25, 21–42.

Besanko, D. and G. Kanatas (1993), "Credit Market Equilibrium with Bank Monitoring and Moral Hazard." *The Review of Financial Studies*, 6, 213–232.

Besanko, D. and A. Thakor (1987), "Collateral and Rationing: Sorting Equilibria in Monopolistic and Competitive Credit Markets." *International Economic Review*, 28, 671–689.

Besley, T., K. Buchardi, and M. Ghatak (2012), "Incentives and the De Soto Effect." *The Quarterly Journal of Economics*, 127, 237–282.

Boot, A. and A. Thakor (1994), "Moral Hazard and Secured Lending in an Infinitely Repeated Credit Market Game." *International Economic Review*, 35, 899–920.

Boot, A., A. Thakor, and G. Udell (1991), "Secured Lending and Default Risk: Equilibrium Analysis, Policy Implications and Empirical Results." *The Economic Journal*, 101, 458–472.

Bottazzi, L., M. Da Rin, and T. Hellmann (2008), "Who Are the Active Investors? Evidence from Venture Capital." *Journal of Financial Economics*, 89, 488–512.

Brick, I. and D. Palia (2007), "Evidence of Jointness in the Terms of Relationship Lending." *Journal of Financial Intermediation*, 16, 427–451.

Cabolis, C., M. Dai, and K. Serfes (2015), "Competition and Specialization in the VC Market: A Non-Monotonic Relationship." Mimeo, Bennett S. LeBow College of Business, Drexel University.

Chakraborty, A. and A. Citanna (2005), "Occupational Choice, Incentives and Wealth Distribution." *Journal of Economic Theory*, 122, 206–224.

Chen, J. (2013), "Estimation of the Loan Spread Equation with Endogenous Bank-Firm Matching." In *Structural Econometrics Models* (E. Choo and M. Shum, eds.), volume 31 of *Advances in Econometrics*, 251–290, Emerald, UK.

Dam, K. (2015), "Job Assignment, Market Power and Managerial Incentives." *The Quarterly Review of Economics and Finance*, 57, 222–233.

Dam, K. and D. Pérez-Castrillo (2006), "The Principal-Agent Matching Market." *Frontiers of Theoretical Economics*, 2, Article 1.

Degryse, H. and P. Van Cayseele (2000), "Relationship Lending within a Bank-Based System: Evidence from European Small Business Data." *Journal of Financial Intermediation*, 9, 90–109.

Dimov, D. and D. Shepherd (2005), "Human Capital Theory and Venture Capital Firms: Exploring 'Home Runs' and 'Strike Outs'." *Journal of Buniness Venturing*, 20, 1–21.

Edmans, A., X. Gabaix, and A. Landier (2009), "A Multiplicative Model of Optimal CEO Incentives in Market Equilibrium." *The Review of Financial Studies*, 22, 4881–4917.

Farrell, J. and S. Sctochmer (1988), "Partnerships." *The Quarterly Journal of Economics*, 103, 279–297.

Hölmstrom, B. and J. Tirole (1997), "Financial Intermediation, Loanable Funds, and the Real Sector." *The Quarterly Journal of Economics*, 112, 663–691.

Legros, P. and A. Newman (2007), "Beauty Is a Beast, Frog Is a Prince: Assortative Matching with Nontransferabilities." *Econometrica*, 75, 1073–1102.

Li, F. and M. Ueda (2012), "Why Do Reputable Agents Work for Safer Firms?" *Finance Research Letters*, 2009, 2–12.

Linton, O., E. Maasoumi, and Y. Whang (2005), "Consistent Testing for Stochastic Dominance under General Sampling Schemes." *The Review of Economic Studies*, 72, 735–765.

Määttänen, N. and M. Terviö (2014), "Income Distribution and Housing Prices: An Assignment Model Approach." *Journal of Economic Theory*, 151, 381–410.

Maskin, E. and J. Riley (1984), "Monopoly with Incomplete Information." *RAND Journal of Economics*, 15, 171–196.

Mehran, H. and A. Thakor (2011), "Bank Capital and Value in the Cross-Section." *The Review of Financial Studies*, 24, 1020–1067.

Mookherjee, D. and D. Ray (2002), "Contractual Structure and Wealth Accumulation." *The American Economic Review*, 92, 818–849.

Repullo, R. and J. Suárez (2000), “Entrepreneurial Moral Hazard and Bank Monitoring: A Model of the Credit Channel.” *European Economic Review*, 44, 1931–1950.

Sattinger, M. (1979), “Differential Rents and the Distribution of Earnings.” *Oxford Economic Papers*, 31, 60–71.

Schwert, M. (2018), “Bank Capital and Lending Relationships.” Forthcoming, *The Journal of Finance*.

Shapley, L. and M. Shubik (1971), “The Assignment Game I: The Core.” *International Journal of Game Theory*, 1, 111–130.

Sorensen, M. (2007), “How Smart Is Smart Money? A Two-Sided Matching Model of Venture Capital.” *The Journal of Finance*, 62, 2725–2762.

Stein, J. (2002), “Information Production and Capital Allocation: Decentralized versus Hierarchical Firms.” *The Journal of Finance*, 62, 1891–1921.

Terviö, M. (2008), “The Difference that CEOs Make: An Assignment Model Approach.” *The American Economic Review*, 98, 642–668.

von Lilienfeld-Toal, U. and D. Mookherjee (2016), “A General Equilibrium Analysis of Personal Bankruptcy Law.” *Economica*, 83, 31–58.