Problem Set: Contract Theory

Problem 1
A risk-neutral principal P hires an agent A, who chooses an effort $a \geq 0$, which results in gross profit $x = a + \varepsilon$ for P, where ε is uniformly distributed on $[0, 1]$. A's payoff equals

$$u(w, a) = \frac{w^{1-\rho}}{1-\rho} - \frac{a^2}{2},$$

where w denotes a non-negative wage paid by P, and $\rho > 0, \neq 1$ is a parameter of (relative) risk-aversion. A has an outside option of $\bar{u}$ which is non-negative if $\rho < 1$ and negative if $\rho > 1$.

(a) If a is contractible, characterize the first-best wage and effort levels.

(b) If a is not contractible, but the profit x is contractible, and $\rho \in (0, 1)$, find a condition on the parameters of the problem (specifically, on ρ) which ensure that the first-best profit can be achieved by P. If $\bar{u} = 0$, when is this condition satisfied?

(c) If $\rho > 1$ what can you say about implementability of the first-best profit when a is not contractible? How would you interpret these results?

Problem 2
Consider a contractual relationship between a landlord (principal) and a tenant (agent) who provide inputs for the production of a single crop. Production requires one plot of land and one unit of labor. The landlord is endowed with one unit of land and the tenant with one unit of labor. The landlord and the tenant are both risk neutral. The tenant chooses two actions, to be referred to as effort and risk, which affect the probability distribution of output ex-ante. These actions are denoted by $e \in [\underline{e}, \bar{e}]$ and $r \in [\underline{r}, \bar{r}]$. None of the actions can be observed by the landlord. The distribution of final output $q \in [\underline{q}, \bar{q}]$ is given by $F(q \mid e, r)$ with the corresponding density $f(q \mid e, r)$ which satisfies MLRP with respect to effort e, i.e,

$$\frac{d}{dq}\left[\frac{f_e(q \mid e, r)}{f(\mid e, r)}\right] \geq 0.$$

The distribution of q is also affected by the choice of risk by the tenant. We assume that the tenant can deliberately increase the riskiness of output in the sense that

$$\int_{\underline{q}}^{\hat{q}} F_r(q \mid e, r) \geq 0 \;\text{ for all }\; \hat{q} \in [\underline{q}, \bar{q}].$$

The tenant incurs a private cost for his actions $\psi(e, r)$. Assume that this function is twice continuously differentiable, monotonically increasing and convex in e and r. Also $\psi(0, 0)$ and $\psi_{er}(e, r) \geq 0$. A tenancy contract is a tuple (R, s) where R is the fixed component and s is the share of output q that accrues to the tenant. That is, for a given level of output q, the tenant earns $sq - R$ and the landlord earns $R + (1-s)q$. Thus, when $R < 0$ and $s = 0$, the contract is a *fixed wage contract*, and when $R > 0$ and $s = 1$, the contract is a *fixed rent contract*. When $R > 0$ and $s \in (0, 1)$ the contract is a *share-cropping contract*. The tenant is subject to limited liability with liability limits equal to 0. Moreover, the tenant's reservation utility is given by $\bar{u} \geq 0$. With the above data in hand, answer the following questions:

1. Formulate the optimal contracting problem between the landlord and the tenant.
2. Solve for the optimal contract when both effect and risk are observable.
3. Solve for the optimal contract when both effort and risk are unobservable. Show that the optimal contract is a share-cropping contract, and the optimal effort $e* < e^{FB}$ and risk $r* \geq r^{FB}$ where e^{FB} and r^{FB} are the first best levels of effort and risk, respectively.
4. Show that when only effort (risk) is observable, the optimal contract entails a fixed wage (rent). Explain this result intuitively.

Problem 3
Consider a variant of the above problem where both the landlord and the tenant supply labor inputs, but there is no choice of risk. Denoted by $a \in [\underline{a}, \bar{a}]$ the effort exerted by the landlord. Assume further that the production function of final output is given by:

$$q = q(e, a) + \varepsilon,$$

where ε is random variable with zero mean and variance σ^2. Make the standard neo-classical assumptions on $q(e, a)$, i.e., $q_e, q_a > 0$, $q_{ee}, q_{aa} < 0$ and $q_{ea} \geq 0$. Let $\psi(e)$ and $\phi(a)$ denote the costs of efforts of the tenant and the principal, respectively. Tenant's reservation utility is given by $\bar{u} \geq 0$. Denote by $w(q)$ the state-contingent transfer the tenant makes to the landlord.

1. Show that the outcome can be implemented by a linear contract of the form $w(q) = R + (1 - s)q$.
2. Solve for the optimal contracts when both efforts are unobservable.
3. Under what conditions the optimal contract entails $s = 0$ and $s = 1$? Interpret your results.

Problem 4
We consider two entrepreneurs each of whom can carry out a project with the following characteristics. Investing 1 generates a stochastic output which can take two values, $z > 0$ or 0. The probability of success (i.e., of getting z) depends in the entrepreneur's effort, e, which can take also two values $\bar{e} > 0$ or 0. The probability of success is $\bar{p}$ for a high level of effort $\bar{e}$ and $\underline{p}$ for no effort with $\bar{p} > \underline{p} > 0$. The disutility of effort is ψ for $\bar{e}$ and zero for $e = 0$. Each entrepreneur has no wealth and must borrow to invest. He can only repay his loan if he succeeds. Denoting by x the entrepreneur's share of output, his expected utility is

$$U(x, e) = \begin{cases} \bar{p}x - \psi & \text{if he exerts effort } \bar{e}, \\ \underline{p}x & \text{if he exerts no effort.} \end{cases}$$

Funds are supplied by a profit-maximizing bank which has a cost of funds r. We assume that

$$\bar{p}z - \psi > r > \underline{p}z$$

1. Determine the optimal contract offer to an entrepreneur when there is a single entrepreneur.
2. There are now two entrepreneurs who do not observe each other's effort level. A group lending contract calls for a payment x when the partner succeeds and y when he fails. Consider a group

lending contract which induces effort of both entrepreneur as a Nash equilibrium. Show that a group lending contract does not perform better than the individual contracts considered in question 1.

3. We suppose now that entrepreneur observe each other's effort level and coordinate their effort levels. Consider the program of the bank which implements effort by both entrepreneurs with group lending contracts:

$$\max_{x\geq 0, y\geq 0} \quad \bar{p}(2z-2x)+\bar{p}(1-\bar{p})(2z-2y)-2r$$

$$\text{subject to} \quad 2\bar{p}x+2\bar{p}(1-\bar{p})y-2\psi \geq 2\underline{p}x+2\underline{p}(1-\underline{p})y, \quad (1)$$

$$2\bar{p}x+2\bar{p}(1-\bar{p})y-2\psi \geq 2\underline{p}\bar{p}x+\underline{p}(1-\bar{p})y+\bar{p}(1-\underline{p})y-\psi, \quad (2)$$

$$2\bar{p}x+2\bar{p}(1-\bar{p})y-2\psi \geq 0. \quad (3)$$

Explain each of the above constraints. Find the optimal contract. Show that it is better than individual contracts for the bank. Explain why.

Problem 5
The Principal Car Sales Company has hired two sales agents $i = 1, 2$. Sales revenue achieved by agent i equals $x_i = a_i + u_i$, where a_i is the effort of agent i, and u_1 and u_2 are normal, i.i.d., zero mean variables with variance $\sigma^2/2$. Both agents are risk-neutral and effort averse, but agent 2 has a lower disutility of effort. So agent 1's payoff is

$$u(w_1, a_1) = w_1 - \frac{a_1^2}{2},$$

while agent 2's is

$$u(w_2, a_2) = w_2 - \frac{\beta a_2^2}{2},$$

where $\beta \in (0, 1)$ and w_i denotes the wage paid to i. Both agents have the same outside option utility of $\bar{u}$. The car company's payoff is $x_1 + x_2 - w_1 - w_2$.

(a) If the car company can monitor effort and contract with the agents based on their observed efforts, derive the (first-best) contracts and effort levels.

(b) Now suppose that the car company cannot monitor effort nor the actual revenues achieved by each agent. It can only observe which agent achieved the highest sales. Find conditions under which there is a symmetric rank order tournament (with a base salary of z and a 'best sales agent' prize of W) which implements the first-best efforts for a Nash equilibrium, in which each agent attains at least his outside option in expectation.

(c) Show that the first-best level of expected profit cannot be attained by the car company with a symmetric tournament.

(d) Can you suggest a modification of the tournament incentive scheme (i.e., based only on ordering of the sales of the two agents) which will implement the first-best.

Problem 6
A monopolist wishes to sell a good produced at constant unit cost c to a large population of consumers

with heterogenous preferences: a consumer of type θ has a payoff $\theta \log q - t$ for consuming $q > 0$ units of the good, and paying t dollars for it. If $q = 0$ then the consumer's payoff is $-t$. The random variable θ is distributed uniformly on $[0, 1]$. The monopolist cannot identify the type of any given consumer.

(a) If $q(\theta)$ denotes the quantity sold to type θ, find a condition on this function $q(\cdot)$ that ensures that it is incentive compatible, i.e., there exists some pricing rule $t(q)$ for which $q(\theta)$ is the optimal purchase of type θ)).

(b) For any such incentive compatible $q(\cdot)$, what is the associated set of payments (i.e., $t(\theta)$) that customers (of type θ) make to the monopolist?

(c) Obtain an expression for total prfit of the monopolist as a function only of the selling strategy $q(\cdot)$.

(d) Calculate the optimal selling strategy $q^*(\theta)$, and the pricing function $t(q)$ which implements it (in the sense of (a) above). (Note: Be careful about corner solutions.)

Problem 7

Consider a risk-neutral government that wants to establish a policy of subsidies to firms that carry out efforts e to decontaminate. The cost to a firm of the decontaminating action is θe^2, where θ is a parameter whose value depends on the type of firm at hand, $\theta \in [1, 2]$. The government's policy consists in a certain decontamination effort e and a transfer t that the firm will receive if decontamination has been carried out. The firm will not accept the subsidy scheme if it does not at least cover the costs. The firm is risk-neutral. The government bears in mind the social benefits of decontamination, valued at $2e$. On the other hand, the government prefers to pay the lowest subsidy possible to the firm. The only source of government's revenue is distortionary taxes. Let λ be the shadow cost of public funds, i.e., if the government wants to collect \$1 it costs her $\$(1+\lambda)$, and hence the social utility of transferring t dollars to the firm is λt (convince yourselves why?).

(a) Calculate the first-best level of decontamination $e^*(\theta)$ and transfer $t^*(\theta)$.

(b) Now assume that the government does not know θ, and believes that it is uniformly distributed on $[1, 2]$. Characterize the optimal mechanism $M = \{e(\theta), t(\theta)\}$.

(c) The government is not always interested in offering a subsidy that all firm types will accept, but rather sometimes it is preferable to leave out those firms with hight decontamination costs. Hence, the government can design a mechanism $M^0 = \{e(\theta), t(\theta) \mid \theta \in [1, \theta^0]\}$ where $\theta^0 \in (1, 2)$. Calculate the optimal mechanism for a fixed θ^0. Prove that it is optimal for the government to design a mechanism intended for all firm types, i.e., $\theta^0 = 2$.

Problem 8

We consider a firm which has a revenue R, but creates a level of pollution x from its activities. The damage created by the level of pollution x is $D(x)$ with $D'(x) > 0$, $D''(x) \geq 0$. The production cost of the firm is $C(x, \theta)$, with $C_x < 0$, $C_{xx} > 0$ and θ is a parameter known only to the firm, which can take two values $\{\bar{\theta}, \underline{\theta}\}$ and $\nu = \Pr(\theta = \bar{\theta})$ is common knowledge.

1. The level of pollution $x^*(\theta)$ corresponding to the complete information optimum is characterized by
$$D'(x) + C_x(x, \theta) = 0.$$

Show that, if the regulator is not obliged to satisfy a participation constraint of the firm, he can implement $x^*(\theta)$ by asking a transfer equal to the cost of damage up to a constant.

2. Suppose now that the firm can refuse to participate (but in this case has a zero utility level) and assume that the regulator has (up to a constant) the following objective function

$$W = -D(x) - (1+\lambda)t + t - C(x, \theta) \tag{4}$$

where t is the transfer from the regulator to the firm and $1+\lambda$ is the opportunity cost of social funds. When the regulator must satisfy the firm's participation constraint

$$t - C(x, \theta) \geq 0 \quad \text{for all} \quad \theta,$$

characterize the decision rule $\hat{x}(\theta)$, $\theta \in \{\bar{\theta}, \underline{\theta}\}$, which maximizes W under complete information. Compare with Problem 1.

3. We assume in addition that $C_\theta < 0$ and $C_{x\theta} < 0$. Determine the menu of contracts $(\underline{t}, \underline{x})$ and $(\bar{t}, \bar{x})$ which maximize the expectation of (1) under participation and incentive constraints of the firm.

4. Same problem when θ is distributed according to the distribution $F(\theta)$ with density $f(\theta)$ on the interval $[\underline{\theta}, \bar{\theta}]$ with

$$\frac{d}{d\theta}\left(\frac{1-F(\theta)}{f(\theta)}\right) < 0, \;\; C_{xx\theta} \geq 0 \;\; \text{and} \;\; C_{\theta\theta x} \leq 0.$$

Problem 9

Consider a simple multitask agency model as in Holmstrom and Milgrom (1991). The economy consists of two risk-neutral individuals: one landlord (L) and one tenant (T). The landlord has two plots of land which differ in quality. The production function of plot $i = 1, 2$ is given by $Q_i = \theta_i e_i$, where $\theta_i \in [\underline{\theta}_i, \bar{\theta}_i]$ with $0 < \underline{\theta}_i < \bar{\theta}_i$ represents the productivity of plot i, and e_i is the labor input or effort required in plot i. Let the joint probability distribution of θ_1 and θ_2 be given by $F(\theta_1, \theta_2)$, and $F_1(\theta_1)$ and $F_2(\theta_2)$ be the respective marginal distributions. We assume that the true realizations of productivity are independent across plots, i.e, $F(\theta_1, \theta_2) = F_1(\theta_1)F_2(\theta_2)$. The tenant has no initial wealth, and incurs cost of efforts which is given by:

$$\psi(e_1, e_2) = \frac{1}{2}\left(c_1 e_1^2 + c_2 e_2^2\right) + \delta e_1 e_2,$$

with $\psi_{ii} = c_i > 0$ for $i = 1, 2$, and $\psi_{12} = \delta \in [0, \sqrt{c_1 c_2})$, which represents the degree of *effort substitution*. Tenant's efforts (e_1, e_2) are not verifiable, and hence there are potential moral hazard problems in effort choice. We normalize the tenant's outside option to 0. A contract $\beta_i = (\alpha_i, R_i)$ for $i = 1, 2$ specifies the tenant's share α_i of output from plot i and the rental payment R_i to be made on plot i. If $\alpha_i = 1$ for $i = 1, 2$ and $R_i > 0$, then the contract is a *pure rent* contract on plot i. On the other hand, share contract emerges when $\alpha_i < 1$ for at least one i. The contracts are subject to limited liability of the the tenant, i.e., when the tenant fails to meet his rental obligations on plot i, the landlord confiscates the entire output from this plot.

(a) Show that when $\delta = 0$, in the second-best outcome, the optimal contract is a pure rent contract.

(b) Show that when $\delta > 0$, in the second-best outcome, the optimal contract is a share contract, i.e., $\alpha_i < 1$ for some $i = 1, 2$. Find a necessary and sufficient condition for the optimality of share contract.