CHAPTER 1: Moral Hazard with Single Agent*

1 Principal-agent problems: symmetric and asymmetric information

Throughout this and the subsequent chapters we will built on the following scenario. There are two classes of individuals in the economy: *principal* who may be an entrepreneur, a landlord, an investor, etc. and *agent* who may be a worker, a manager, a tenant, an entrepreneur, etc. When a principal and an agent enter into a contractual relationship, the agent has private information over the actions he may take (moral hazard) or over his type such as productivity, efficiency, etc. (adverse selection, see Chapter 3) which influence the (random) performance of the relationship. For instance, an entrepreneur hires a manager to work in her firm. The firm's performance (say, profit) depends on the effort made by the manager. When the effort levels can be observed by the principal, we are in a situation of *symmetric information*. On the other hand, when the effort choice of the manager cannot be verified by the principal, we are in a world characterized by *asymmetric information*.

Our objective is to determine the optimal managerial compensation under both situations. In general, under symmetric information, the optimal compensation is determined by solving a Pareto program where the principal maximizes her net expected payoff subject to the constraint that the compensation package guarantees a minimum expected payoff to the agent. Such constraints are called the *individual rationality* or *participation* constraint of the agent. This situation is equivalent to maximizing the aggregate surplus of the relationship, and is called the *first best* situation. When the manager's effort choice cannot be verified, a compensation contract cannot be based on the managerial effort. In addition to the participation constraint, one then needs to impose the *incentive compatibility* constraint which implies that although the effort of the agent cannot be contracted upon, the principal can anticipate that the agent will choose the effort level following some optimizing behavior. The optimal contracts obtained in this situation are called the *second best* contracts.

2 Moral hazard

Consider a contracting problem between a principal and an agent where the principal hires the agent to accomplish a task. The agent chooses an action e, e.g. effort, investment, production technique, etc. which influences the performance q of the task. The principal only cares about the performance. Effort is costly for the agent and he requires to be compensated. When effort is non-verifiable, the best the principal can do is to relate compensation to performance. Such compensation scheme entails a loss of

*These notes are heavily borrowed from (Bolton and Dewatripont, 2005, Chapter 4).

efficiency since performance is only a noisy signal of effort. Such situations give rise to what is called a *moral hazard* problem. This is a very simplified scenario – the real world situation may be more complex. In general, a principal-agent relationship faces a moral hazard problem if the parties are unable to contract upon some decision variable(s). Following are some examples of moral hazard problem.

Example 1 A landlord (principal) hires a tenant (agent) to cultivate a plot of land. The crop yield depends on the effort e exerted by the tenant and a random variable ε:

$$q = e + \varepsilon,$$

where $\varepsilon \sim N(0, \sigma^2)$. The tenant's effort e is not verifiable. Effort is costly to the tenant. Let $\psi(e)$ is the cost function of effort which is strictly increasing and convex in e. The tenant faces the following trade-off. An increase in the effort level increases q, but it also raises his disutility. ■

Example 2 A firm's (principal) constant marginal cost of production θ may take a high value θ_H or a low value θ_L. Initially, the firm is inefficient, i.e., $\theta = \theta_H$, and hires a manager (agent) to carry out R&D activities to make the production process more efficient, i.e., reduce θ to the lower level θ_L. Let the manager chooses an R&D effort e which lowers the firm's marginal cost with a probability $p(e)$ such that $p''(e) < 0 < p'(e)$. The effort cost function is given by $\psi(e) = e$. Effort is not observable by the firm. ■

Example 3 A penny less startup firm (agent) has a setup cost of \$1 which must be raised from an external investor (principal). After the investor agrees to invest, the firm may choose one of the two projects P^1 and P^2. Choice of project is not verifiable by the investor. Each project P^i has a verifiable income y_i and a non-verifiable private benefit b_i. Assume that $y_1 > y_2$ and $b_1 < b_2$, i.e., from the viewpoint of the investor P^1 is a good project and P^2 is a bad project. By incurring a non-verifiable monitoring cost $\phi(m)$ with ϕ', $\phi'' > 0$ the investor can make sure that P^1 is undertaken with probability m. There are moral hazard problems both in the choices of project and monitoring. ■

3 Timing of events

The principal-agent relationship lasts for five dates, $t = 0, 1, 2, 3, 4$. At date 0, the principal offers a contract. At $t = 1$, the agent accepts or rejects the contract. At $t = 2$, the agent exerts effort. At $t = 3$, the output is realized. Finally, at date 4 the contract is executed.

4 Optimal contracts with two performance outcomes

Consider a situation where performance may take only two values: $q \in \{q_H, q_L\}$ with $q_H > q_L \geq 0$. When $q = q_H$ the agent's performance is a "success", and it is a "failure" otherwise. Let the probability of success be given by Prob.$[q = q_H \mid e] = p(e)$ with $p''(e) \leq 0 < p'(e)$ where $e \in [0, 1]$ is the effort chosen by the agent. Assume that $p(0) = 0$ and $p'(0) > 1$. The principal's utility function is given by $V(q - w)$ with $V'' \leq 0 < V'$, and the agent has utility function $U(w, e) = u(w) - \psi(e)$ with $u'' \leq 0 < u'$, $\psi' > 0$, and $\psi'' \geq 0$. Assume that $\psi(e) = e$.

4.1 First-best contracts

Suppose that the agent's effort choice is verifiable, and hence the compensation scheme can be made contingent on this choice. Let the compensation scheme be $w_\theta = w(q_\theta)$ for $\theta = H, L$. Then the principal solves the following maximization problem:

$$\max_{\{e, w_H, w_L\}} \; p(e)V(q_H - w_H) + [1 - p(e)]V(q_L - w_L) \tag{M$_1$}$$

$$\text{subject to} \quad p(e)u(w_H) + [1 - p(e)]u(w_L) - e \geq \bar{u}, \tag{IR}$$

where $\bar{u}$ is the agent's outside option. Assume without loss of generality that $\bar{u} = 0$. Let λ be the Lagrange multiplier associated with (IR). The first order conditions with respect to w_H and w_L yield the following optimal coinsurance, or so-called *Broch rule*:

$$\frac{V'(q_H - w_H)}{u'(w_H)} = \lambda = \frac{V'(q_L - w_L)}{u'(w_L)}. \tag{1}$$

Given that $V' > 0$ and $u' > 0$, we have $\lambda > 0$, i.e., the (IR) constraint binds at the optimum. The first order condition with respect to effort is given by:

$$p'(e)[V(q_H - w_H) - V(q_L - w_L)] + \lambda p'(e)[u(w_H) - u(w_L)] = \lambda, \tag{2}$$

which, together with the Borch rule, determines the optimal effort e. Why Borch rule implies Pareto optimality? Let $x_\theta := q_\theta - w_\theta$, the principal's income at state $\theta = H, L$. Notice that the slopes of the indifference curves of the principal and the agent are respectively given by:

$$\frac{dx_L}{dx_H} = -\frac{p(e)}{1 - p(e)} \frac{V'(x_H)}{V'(x_L)},$$
$$\frac{dw_L}{dw_H} = -\frac{p(e)}{1 - p(e)} \frac{u'(w_H)}{u'(w_L)}.$$

Hence, (1) implies tangency of the indifference curves, i.e., Pareto optimality. Notice a few important properties of the set of Pareto optimal solutions.

(a) At the optimum, it is always the case that (i) $x_H \geq x_L$ and (ii) $w_H \geq w_L$. To show (i), suppose on the contrary that $x_H < x_L$. This inequality implies that $w_H - w_L > q_H - q_L > 0$. On the other hand, since $V''(\cdot) \leq 0$, we have $V'(x_H) \geq V'(x_L)$ for $x_H < x_L$. Then it follows from the Borch rule that $u'(w_H) \geq u'(w_L)$ which implies $w_H < w_L$ which is a contradiction. In a similar fashion one can prove (ii). Also, conditions (i) and (ii) together imply $0 \leq x_H - x_L \leq q_H - q_L$ and $0 \leq w_H - w_L \leq q_H - q_L$. Full insurance occurs when either (a) $w_H - w_L = 0$ and $x_H - x_L = q_H - q_L$, or (b) $x_H - x_L = 0$ and $w_H - w_L = q_H - q_L$. What do conditions (i) and (ii) mean. Consider an Edgeworth box for this principal-agent economy in which the horizontal axes represent incomes at state H and the vertical axes stand for those in state L. This property of the optimum says that the Pareto set lies between the two 45^0 lines, the full insurance lines for the principal and the agent respectively. Clearly, due to optimal risk sharing motives, both parties cannot be fully insured at the same time as long as $q_H > q_L$.

(b) Principal is risk neutral, i.e., $V(x) = x$. Then (1) implies that $w_H = w_L = w^*$, i.e., the optimum entails full insurance for the agent. And the optimal effort level e^* is given by:

$$u(w^*) = e^*.$$

Notice that, at the optimum, $p'(e^*)(q_H - q_L) = 1/u'(w^*)$, i.e., the marginal productivity of effort is equal to the marginal cost the the principal incurs to compensate the agent for his effort cost. In this case we have $x_H - x_L = q_H - q_L$, i.e., the risk neutral principal assumes the entire risk of output measured by $q_H - q_L$.

(c) Agent is risk neutral, i.e., $u(x) = x$. Then the optimal entails full insurance for the principal with $x_H = x_L = x^*$ and $w_H - w_L = q_H - q_L$. All risks due to output fluctuations are absorbed by the risk neutral agent.

4.2 Second-best contracts

When the agent's effort choice cannot be verified by the principal, the compensation contracts cannot be made contingent upon the agent's choice of effort, and the first-best contracts cannot be implemented. Then the agent's performance-contingent compensation induces him to choose an effort level e that maximizes his expected payoff:

$$p(e)u(w_H) + [1 - p(e)]u(w_L) - e \geq p(\hat{e})u(w_H) + [1 - p(\hat{e})]u(w_L) - \hat{e} \quad \text{for all } \hat{e} \in [0, 1]. \tag{IC}$$

The above constraint is called the *incentive compatibility constraint*. The first order condition of the maximization problem is given by:

$$p'(e)[u(w_H) - u(w_L)] = 1. \tag{IC$'$}$$

Since $p'(\cdot) > 0$, (IC$'$) implies that $u(w_H) > u(w_L)$, which in turn implies that $w_H > w_L$ since $u'(\cdot) > 0$. Let $\Delta \equiv u(w_H) - u(w_L)$. Then, from (IC$'$) it follows that

$$\frac{de}{d\Delta} = -\frac{p'(e)}{p''(e)\Delta} > 0. \tag{3}$$

Now, consider an increase in $w_H - w_L$, which can be achieved either (i) by increasing w_H and keeping w_L constant, or (ii) by decreasing w_L and keeping w_H constant, or (iii) by increasing w_H and decreasing w_L. Since $u'(\cdot) > 0$, in all the three cases Δ must increase. Therefore,

Lemma 1 *At any incentive compatible contract, i.e., any (w_L, w_H) that satisfies (IC$'$), we must have*

(a) $w_H > w_L$;

(b) *effort e is increasing in $w_H - w_L$.*

The optimal second-best contracts (w_H, w_L, e) are determined by maximizing (M_1) subject to (IR) and (IC$'$). Let μ be the Lagrange multiplier associated with the incentive compatibility constraint. The first order conditions with respect to w_H and w_L are respectively given by:

$$\frac{V'(q_H - w_H)}{u'(w_H)} = \lambda + \mu \left[\frac{p'(e)}{p(e)}\right], \quad (4)$$

$$\frac{V'(q_L - w_L)}{u'(w_L)} = \lambda - \mu \left[\frac{p'(e)}{1 - p(e)}\right]. \quad (5)$$

When $\mu = 0$, we obtain the Borch rule, and $(w_L, w_H) = (w_L^*, w_H^*)$ where $*$ denotes the first-best contracts. Since $\mu \geq 0$ and $p'(e) > 0$, (4) and (5) imply that

$$\frac{V'(q_H - w_H)}{u'(w_H)} \geq \frac{V'(q_H - w_H^*)}{u'(w_H^*)}, \quad \text{and} \quad \frac{V'(q_L - w_L)}{u'(w_L)} \leq \frac{V'(q_L - w_L^*)}{u'(w_L^*)}.$$

Notice that, given $V'' \leq 0$ and $u'' \leq 0$, the following function

$$H(w) := \frac{V'(q - w)}{u'(w)}$$

is increasing in w, and hence the above two inequalities imply that

Lemma 2 *The second best contract has the property that $w_H \geq w_H^*$ and $w_L \leq w_L^*$, and hence, $w_H - w_L \geq w_H^* - w_L^*$.*

In Section 5, we will prove that, at the optimum, $\mu > 0$ under very general conditions, and hence, the contract is distorted from the optimal risk sharing contract implied by the Borch rule. Moreover, from Lemma 1 we have $w_H > w_L$. Notice that, even if the principal is risk neutral, insuring fully the agent is not optimal. This occurs because, as opposed to the first-best situation, the principal faces a trade-off between risk sharing and incentive provision. It easily follows from the incentive constraint that the effort exerted by the agent is increasing in $w_H - w_L$ since $u' > 0$. Hence, in order to induce the agent to exert the highest possible incentive compatible effort, the principal would like to make the difference $w_H - w_L$ as large as possible. Clearly, full insurance would undermine such incentives. We will analyze further properties of the optimal second-best contracts in more general setting in the next section.

5 A general model of incentive contracts

Following Hölmstrom (1979), we now consider a general nonlinear incentive scheme. Let the firm's output q depends on some random variable describing possible contingencies and the effort of the agent, i.e., $q = q(\theta, e)$ where $\theta \in \Theta$ is the random variable and $e \in A \subseteq \mathbb{R}$. We assume that both Θ and A are compact sets. The principal may be risk averse and has a utility function $V(q - w)$ with $V'(\cdot) > 0$ and $V''(\cdot) \leq 0$. The agent is risk averse and has a separable utility function $u(w) - \psi(e)$ with $u'(\cdot) > 0$, $u''(\cdot) \leq 0$, $\psi'(\cdot) > 0$ and $\psi''(\cdot) \geq 0$. Let the conditional distribution and density of output are respectively given by $F(q \mid e)$ and $f(q \mid e)$ over the support $[q_{min}, q_{max}]$. Then the principal's problem can be written as

$$\max_{\{e,w(q)\}} \int_{q_{min}}^{q_{max}} V(q-w(q))f(q \mid e)dq \tag{M$_2$}$$

$$\text{subject to} \quad \int_{q_{min}}^{q_{max}} u(w(q))f(q \mid e)dq - \psi(e) \geq \bar{u}, \tag{IR}$$

$$e \in \text{argmax}_{\hat{e}} \left\{ \int_{q_{min}}^{q_{max}} u(w(q))f(q \mid \hat{e})dq - \psi(\hat{e}) \right\}. \tag{IC}$$

The first and second order conditions of the agent's maximization problem are given by:

$$\int_{q_{min}}^{q_{max}} u(w(q))f_e(q \mid e)dq - \psi'(e) = 0, \tag{IC$_a$}$$

$$\int_{q_{min}}^{q_{max}} u(w(q))f_{ee}(q \mid e)dq - \psi''(e) < 0. \tag{IC$_b$}$$

As we have been doing in the previous sections, we will replace (IC) by (IC$_a$) leaving aside for the time being the constraint (IC$_b$). The Lagrangean is therefore given by:

$$\mathscr{L} = \int_{q_{min}}^{q_{max}} \left[V(q-w(q))f(q \mid e) + \lambda \left\{ u(w(q))f(q \mid e) - \psi(e) - \bar{u} \right\} + \mu \left\{ u(w(q))f_e(q \mid e) - \psi'(e) \right\} \right] dq$$

The first order necessary condition with respect to $w(q)$ is given by:

$$\frac{V'(q-w(q))}{u'(w(q))} = \lambda + \mu \frac{f_e(q \mid e)}{f(q \mid e)} \quad \text{for all } q. \tag{6}$$

5.1 First best contracts

The first best contract is implemented when the agent's effort can be verified by the principal. In this case any effort level can be implemented, and the incentive compatibility constraint (IC) can be ignored. This is equivalent to a situation where $\mu = 0$. Then the first order condition (6) reduces to the following Borch rule:

$$\frac{V'(q-w(q))}{u'(w(q))} = \lambda > 0 \quad \text{for all } q. \tag{7}$$

Now we study the shape of the compensation schedule. Differentiating (7) with respect to q we get

$$-V''(1-w'(q)) + \lambda u'' w'(q) = -V''(1-w'(q)) + \frac{V'}{u'} u'' w'(q) = 0. \tag{8}$$

Define by $\eta_P = -V''/V'$ and $\eta_A = -u''/u'$ the coefficients of absolute risk aversion of the principal and the agent, respectively. Then Condition (8) becomes

$$w'_{FB}(q) = \frac{\eta_P}{\eta_P + \eta_A}. \tag{9}$$

Notice that the coefficients of risk aversion are not constant in general, and hence the equilibrium compensation function $w(q)$ is nonlinear.

5.1.1 Risk neutral principal

Assume that $V'(\cdot) = \text{constant}$, i.e., the principal is risk neutral. Then Condition (7) requires that $u'(w(q)) =$ constant for all q, i.e., $u'(w(q)) = u'(w(q'))$ for $q \neq q'$. This occurs only if $w(q) = w(q') = w_{FB}$. Therefore, at the optimum the agent receives a fixed salary independent of the output of the firm. In other words, the principal assumes all the output risk and fully insures the agent against the variability of the output. Since $\lambda > 0$, the individual rationality constraint binds at the optimum, and hence

$$w_{FB} = u^{-1}\left(\bar{u} + \psi(e_{FB})\right), \tag{10}$$

where e_{FB} is the effort implemented at the compensation scheme w_{FB}.

5.1.2 Risk neutral agent

Now suppose that the agent is risk neutral instead. Then Condition (7) requires that $V'(q - w(q)) = V'(q' - w(q'))$ for $q \neq q'$. This is true only if $q - w(q) = q' - w(q')$ for any $q \neq q'$. In this case, the principal is fully insured as she sells the firm to the agent at a fixed price $q - w(q)$ and the agent becomes the residual claimant of the firm output. This type of contracts are called *franchise* contracts.

5.1.3 Risk averse principal and agent

When both the principal and the agent are risk averse, the optimal first best contract is characterized by Condition (6), and hence the optimal marginal compensation is given by:

$$w'_{FB}(q) = \frac{\eta_P}{\eta_P + \eta_A} \in (0, 1).$$

The above equation implies that following an improvement in the firm's performance, the agent receives only a part of the increased output as increased compensation. Thus, both the principal and the agent are partially insured against risk. The more risk averse is the agent, i.e., the greater is η_A, the less the firm's performance influences his compensation.

5.2 Second best contracts

It is intuitive that the principal would always like the agent to exert higher effort, and possibly, reward him for not being lazy. But the problem is that since effort is not verifiable, it cannot be contracted upon and the compensation of the agent can only be based on the firm's performance q. First important question is whether high effort is at all desirable as rewarding a very high effort level may be highly costly for the principal. We will assume that $F_e(q \mid e) \leq 0$ with strict inequality for some q. This is to say that given two effort levels $e < e'$, the distribution $F(q \mid e')$ *first order stochastically dominates* the distribution $F(q \mid e)$. Intuitively, in the two-performance case, i.e., $q \in \{q_H, q_L\}$ with $q_H > q_L$, Prob.$[q = q_H \mid e'] >$ Prob.$[q = q_H \mid e]$ if $e' > e$, i.e., higher effort makes 'success' more likely. In other words, higher effort shifts the density $f(q \mid e)$ to the right. The following is an important result related to first order stochastic dominance.

Lemma 3 *Let $h : [q_{min}, q_{max}] \to \mathbb{R}$ is an increasing function, and $F_e(q \mid e) \leq 0$ for all e. Then for any two effort levels e and e' with $e > e'$, we have*

$$E[h(q) \mid e] \geq E[h(q) \mid e'].$$

In particular, $E[q \mid e] \geq E[q \mid e']$.

Proof. Notice that

$$E[q \mid e] - E[q \mid e'] = \int_{q_{min}}^{q_{max}} h(q)[f(q \mid e) - f(q \mid e')]dq = -\int_{q_{min}}^{q_{max}} h'(q)[F(q \mid e) - F(q \mid e')]dq.$$

The above expression is positive since $h'(q) \geq 0$ and $F(q \mid e) \leq F(q \mid e')$. □

First order stochastic dominance is one of the many popular stochastic orders e.g. second order stochastic dominance, hazard rate dominance, likelihood ratio dominance, etc. See (Krishna, 2002, Appendix B) which provides a nice discussion on stochastic orders.

The optimal second best contracts are characterized by Condition (6) along with the binding individual rationality constraint (since $\lambda > 0$). When $\mu = 0$ we are back to the first best and (6) represents the pure risk sharing role of the contract, given by the Borch rule, which is meant to coinsure both the principal and the agent. When $\mu > 0$, the optimal contract, in addition to its risk sharing motive, is an instrument used for incentive provision. Thus, $\mu > 0$ implies the so-called tradeoff between risk sharing and incentive. We will first prove that $\mu > 0$ under very general conditions.

Lemma 4 (Hölmstrom, 1979) *Suppose that the principal has a strictly increasing utility function, i.e., $V' > 0$, and $F_e(q \mid e) \leq 0$. Then $\mu > 0$.*

Proof. Suppose on the contrary that $\mu \leq 0$. The first order condition of the principal's maximization problem with respect to e is given by:

$$\int_{q_{min}}^{q_{max}} \left[V(q - w(q))f_e(q \mid e) + \lambda \left\{u(w(q))f_e(q \mid e) - \psi'(e)\right\} + \mu \left\{u(w(q))f_{ee}(q \mid e) - \psi''(e)\right\}\right] dq = 0.$$

Using the optimality conditions of the agent's maximization problem (IC_a) and (IC_b), note that $\mu \leq 0$ is equivalent to

$$\int_{q_{min}}^{q_{max}} V(q - w(q))f_e(q \mid e)dq \leq 0. \tag{11}$$

call the first best compensation schedules of the principal and the agent $x_{FB}(q)$ and $w_{FB}(q)$, respectively, which are, by Borch rule, are non-decreasing in q with slopes everywhere less than 1. Now consider two subsets $Q_+ := \{q \mid f_e(q \mid e) > 0\}$ and $Q_- := \{q \mid f_e(q \mid e) < 0\}$ of $[q_{min}, q_{max}]$. Notice that, when $\mu \leq 0$, $w(q) \leq w_{FB}(q)$ for all $q \in Q_+$, and $w(q) \geq w_{FB}(q)$ for all $q \in Q_-$. Therefore,

$$\int_{q_{min}}^{q_{max}} V(q - w(q))f_e(q \mid e)dq \geq \int_{q_{min}}^{q_{max}} V(q - w_{FB}(q))f_e(q \mid e)dq. \tag{12}$$

However, since $F_e(q_{min} \mid e) = F_e(q_{max} \mid e) = 0$ for all e, integration by parts implies

$$\int_{q_{min}}^{q_{max}} V(q - w_{FB}(q)) f_e(q \mid e) dq = -\int_{q_{min}}^{q_{max}} V'(q - w_{FB}(q))[1 - w'_{FB}(q)] F_e(q \mid e) dq \tag{13}$$

Since $V' > 0$, $w'_{FB}(q) < 1$ and $F_e(q \mid e) \leq 0$ for all q, the above expression is strictly positive, and hence (12) implies that

$$\int_{q_{min}}^{q_{max}} V(q - w(q)) f_e(q \mid e) dq > 0,$$

which contradicts condition (11). Therefore, μ must be strictly positive at the second best optimum. □

For simplicity assume that the principal is risk neutral, i.e., $V' = 1$, Condition (6) can be written as

$$\frac{1}{u'(w(q))} = \mu + \lambda \frac{f_e(q \mid e)}{f(q \mid e)} \quad \text{for all } q. \tag{6'}$$

Ideally, we would like the principal to pay the agent more for better performance, i.e., the compensation function is monotone. It is easy to show from the above that $w'(q) > 0$ if the following *monotone likelihood ratio property* holds for all q:

$$\frac{d}{dq}\left[\frac{f_e(q \mid e)}{f(q \mid e)}\right] > 0, \tag{MLRP}$$

The MLR property of the density function means that a good performance is a signal that, with high probability, high effort was exerted. Notice that (MLRP) implies first order stochastic dominance, i.e., $F_e(q \mid e) < 0$.

Lemma 5 *The monotone likelihood ratio property implies first order stochastic dominance.*

Proof. Notice that (MLRP) implies for any $q_2 > q_1$ and $e_2 > e_1$,

$$\begin{aligned} &\frac{f(q_2 \mid e_2)}{f(q_2 \mid e_1)} > \frac{f(q_1 \mid e_2)}{f(q_1 \mid e_1)} \\ \iff &f(q_2 \mid e_2) f(q_1 \mid e_1) > f(q_1 \mid e_2) f(q_2 \mid e_1) \end{aligned} \tag{$\star$}$$

The above inequality implies that

(a)

$$\begin{aligned} &\int_{q_{min}}^{q_2} f(q_2 \mid e_2) f(q_1 \mid e_1) dq_1 > \int_{q_{min}}^{q_2} f(q_1 \mid e_2) f(q_2 \mid e_1) dq_1 \\ \iff &f(q_2 \mid e_2) F(q_2 \mid e_1) > f(q_2 \mid e_1) F(q_2 \mid e_2). \end{aligned}$$

Letting $q_2 = q$ we get from the above that

$$\frac{f(q \mid e_2)}{f(q \mid e_1)} > \frac{F(q \mid e_2)}{F(q \mid e_1)} \tag{14}$$

(b) Similarly, integrating ($\star$) with respect to q_2 over $[q_1, q_{max}]$ and letting $q_1 = q$, we get

$$\frac{f(q \mid e_2)}{f(q \mid e_1)} < \frac{1-F(q \mid e_2)}{1-F(q \mid e_1)} \tag{15}$$

Therefore, (14) and (15) together imply

$$\begin{aligned} &\frac{1-F(q \mid e_2)}{F(q \mid e_2)} > \frac{1-F(q \mid e_1)}{F(q \mid e_1)} \\ \iff &F(q \mid e_2) < F(q \mid e_1) \quad \text{for any} \quad e_2 > e_1. \end{aligned} \tag{16}$$

Let $e_2 = e_1 + h$ for $h > 0$. The the above inequality implies that

$$\frac{F(q \mid e_1 + h) - F(q \mid e_1)}{h} < 0.$$

Since $F(q \mid \cdot)$ is continuous, taking the limit of the above as $h \to 0$ we get $F_e(q \mid e) < 0$ which is equivalent to first order stochastic dominance of $F(\cdot \mid e)$. □

FOSD simply means that a higher effort level shifts probability mass to better performance. Notice that the condition (MLRP) is a stronger condition that the FOSD. Following example shows that $F_e(q \mid e) < 0$, but the corresponding density function does not satisfy (MLRP), and hence the optimal wage schedule is non-monotone.

Example 4 (Non-monotone compensation function) Suppose there are only three possible performance realizations: $q_L < q_M < q_H$, and that the agent has two possible effort levels: $e' < e''$. The conditional densities are given in the following table:

	$f(q_L \mid e)$	$f(q_M \mid e)$	$f(q_H \mid e)$
e'	0.5	0.5	0.0
e''	0.4	0.1	0.5

Here the second best compensation function is such that $w(q_H) > w(q_L) > w(q_M)$. The point is that when $q = q_L$ it is almost likely that the agent chose e' as e''. Therefore, the principal does not want to punish the agent too much for low performance realizations. Notice in this example that the corresponding distribution function satisfies FOSD, but this property is not enough to guarantee a monotone compensation function. ■

In order to solve the principal's utility maximization problem we have ignored the constraint (IC_b). This can be done only if (IC_a) is necessary and sufficient for the agent's maximization problem (IC). We know that this would be the case if the agent's objective function is strictly concave in e which is not true in general under any density function $f(\cdot \mid e)$. We therefore require to give additional restriction on the distribution of firm output. We omit such technicalities which can be found in (Bolton and Dewatripont, 2005, Chapter 4).

5.3 Informative signals

Now suppose that $s \in [s_{min}, s_{max}]$ be a signal, which in addition to q, is observed by both parties and hence can potentially be used in constructing the compensation scheme. Under what conditions the principal must condition the incentive scheme both on q and s? If the contract includes s in addition to q, i.e., $w(q, s)$ is an optimal contract, and if both the principal and the agent are strictly better off by using such rules instead of $w(q)$, then the signal s is said to be *valuable*. First consider the following definition.

Definition 1 *Let $f(q, s \mid e)$ be the joint density of q and s. Performance q is a "sufficient statistic" for $\{q, s\}$ with respect to e, or s is "non-informative" about e given q if and only if f is multiplicatively separable in s and e, i.e.,*

$$f(q, s \mid e) = g(q, s)h(q \mid e).$$

The signal s is "informative" about e whenever q is not a sufficient statistic for $\{q, s\}$ with respect to e.

In the context of moral hazard, Hölmstrom (1979) proves the following important result, which can be extended to more general settings.

Proposition 1 *Let $w(q)$ be an optimal compensation scheme for which the agent's choice of effort is unique and interior in $[0, 1]$. Then there exists a compensation scheme $w(q, s)$ which strictly Pareto dominates $w(q)$ if and only if s is informative about e, i.e., a signal is valuable if and only if it is informative.*

Proof. We only provide a sketch of the proof. For details see Hölmstrom (1979). The first order condition for the determination of $w(q, s)$ is given by:

$$\frac{V'(q - w(q, s))}{u'(w(q, s))} = \lambda + \mu \frac{f_e(q, s \mid e)}{f(q, s \mid e)} \quad \text{for all } (q, s). \tag{17}$$

Notice that the likelihood ratio f_e/f is independent of s if and only if s is non-informative about e, and hence the optimal compensation scheme should be independent of the signal s. □

The above result asserts that when q is a sufficient statistic for $\{q, s\}$ with respect to e, i.e., q contains all relevant information about the agent's effort, then there is no additional gain for the principal if she uses the signal s. Conversely, if there is an additional signal that does contain additional statistical information about the agent's effort, then such informative signal must be included in the optimal compensation contract.

6 Optimality of linear contracts

The simplest form of contracts under moral hazard are linear compensation schemes which are prevalent in many principal-agent relationships. For example, a tenant generally pays a fixed rent to his landlord and gets a fixed share of the crop. Franchise contracts are often linear contracts. In what follows we show that under mild restrictions on the utility function of the agent and normality of the distribution of the performance, the optimal contracts are linear. Let the performance q is given by:

$$q = e + \varepsilon, \tag{18}$$

where ε is normally distributed with zero mean and variance σ^2. The agent has constant absolute risk averse (CARA) risk preferences represented by the following utility function:

$$u(w, e) = -\exp\{-\eta[w - \psi(e)]\} \tag{19}$$

where w is the amount of monetary compensation and $\eta > 0$ is the agent's coefficient of absolute risk aversion. For simplicity assume that $\psi(e) = (1/2)ce^2$ with $c > 0$. Consider the following linear contract.

$$w = f + bq \tag{20}$$

where f is the fixed salary and b is the bonus or the "piece rate" related to performance. Notice that the expectation and variance of the agent's income are given by:

$$\begin{aligned} E[w] &= f + be, \\ Var(w) &= b^2\sigma^2. \end{aligned}$$

A risk neutral principal thus solves the following maximization problem:

$$\max_{e, f, b} E[q - w] \tag{M$_3$}$$

$$\text{subject to} \quad E[-\exp\{-\eta[w - \psi(e)]\}] \geq u(\bar{w}), \tag{IR}$$

$$e \in \operatorname{argmax}_{e'} E[-\exp\{-\eta[w - \psi(e')]\}]. \tag{IC}$$

The first constraint (IR) is the *individual rationality* or *participation* constraint which guarantees a minimum expected utility, called the *reservation utility*, equal to $u(\bar{w})$ where $\bar{w}$ denotes the minimum acceptable certain monetary equivalent of the agent's compensation contract. The second is the *incentive compatibility* constraint which asserts that the agent chooses the effort level that maximizes his expected utility. Notice that

$$\begin{aligned} &E[-\exp\{-\eta[w - \psi(e)]\}] \\ =&E[-\exp\{-\eta[f + be + b\varepsilon - (1/2)ce^2]\}] \\ =&-\exp\{-\eta[f + be - (1/2)ce^2]\}E[\exp\{-\eta b\varepsilon\}]. \end{aligned} \tag{21}$$

Since $\varepsilon \sim N(0, \sigma^2)$, its moment generating function given by $E[t\varepsilon] = \exp\{(1/2)\sigma^2 t^2\}$. Therefore, Equation (21) can be written as

$$E[-\exp\{-\eta[w - \psi(e)]\}] = -\exp\{-\eta\hat{w}(e)\}, \tag{22}$$

where

$$\hat{w}(e) = f + be - \frac{\eta}{2}b^2\sigma^2 - \frac{1}{2}ce^2 = E[w] - \frac{\eta}{2}Var(w) - \psi(e)$$

is the *certainty equivalent compensation* of the agent, which is equal to his expected compensation net of his effort cost and of a risk premium. Because of the exponential form of the utility function the agent's maximization problem reduces to:

$$e = \text{argmax}_{\hat{e}} \left\{ f + b\hat{e} - \frac{\eta}{2} b^2 \sigma^2 - \frac{1}{2} c\hat{e}^2 \right\} = \frac{b}{c}. \tag{IC$'$}$$

Substituting the above into the principal's objective function and the individual rationality constraint of the agent, the maximization problem of the principal reduces to:

$$\max_{f,b} \frac{b}{c} - \left[f + \frac{b^2}{c} \right] \tag{M$_3'$}$$

$$\text{subject to} \quad f + \frac{b^2}{2c} - \frac{\eta}{2} b^2 \sigma^2 = \bar{w}. \tag{IR$'$}$$

Solving the above, and using $e = b/c$ we get

$$b^* = \frac{1}{1 + \eta c \sigma^2}.$$

Thus, optimal piece rate decrease when c (marginal cost of effort), η (risk aversion) and σ^2 (randomness of performance) go up. The effect of an increase in σ^2 on b^* implies a trade-off between risk sharing and incentive.

7 Contracts under risk neutrality and limited liability

7.1 Managerial compensation contracts

Consider a contracting problem between a firm (principal) and a manager (agent). The constant marginal cost of production θ of the firm may take two values: 'high' and 'low', i.e., $\theta \in \{\theta_L, \theta_H\}$ with $\theta_H > \theta_L \geq 0$. Initially, the firm is 'inefficient', i.e., it has a marginal cost equal to θ_H. The principal hires a manager whose principal task is to exert R&D effort e in order to reduce the marginal cost to the lower level. Let Prob.$[\theta = \theta_L \mid e] = p(e)$. Assume that $p(0) = 0$ and $p'(e) > 0 \geq p''(e)$ for all $e > 0$. The incentive scheme is given by (f, b) where f is the fixed salary and b is the bonus for cost reduction from θ_H to θ_L. Both the principal and the agent are risk neutral. The principal's maximization problem is given by:

$$\max_{e,f,b} p(e)\pi(\theta_L) + [1 - p(e)]\pi(\theta_H) - [f + p(e)b] \tag{M$_4$}$$

$$\text{subject to} \quad [f + p(e)b] - \psi(e) \geq \bar{u}, \tag{IR}$$

$$e = \text{argmax}_{\hat{e}} [f + p(\hat{e})b] - \psi(\hat{e}), \tag{IC}$$

$$f + b \geq l, \quad \text{and} \quad f \geq l, \tag{LL}$$

where $l \geq 0$ is the agent's liability limit. Constraints (LL) is the agent's *limited liability* constraints that guarantee a minimum final income l at each state of the nature. Using the first order condition of the agent's maximization problem, (IC) can be written as

$$b = \frac{\psi'(e)}{p'(e)}. \tag{IC$'$}$$

Notice that $f \geq l$, and $p'(e) > 0$ and $\psi'(e) > 0$ imply $f + b \geq l$, and hence the first limited liability constraint can be ignored. Also define by $\pi := \pi(\theta_H) - \pi(\theta_L)$ the marginal benefit of cost reduction for the firm. Thus, substituting for b in the principal's objective function and in the constraints, the maximization problem reduces to:

$$\max_{e,f} \; p(e)\pi + \pi(\theta_H) - f - \frac{p(e)\psi'(e)}{p'(e)} \tag{M$_4'$}$$

$$\text{subject to} \quad f + \frac{p(e)\psi'(e)}{p'(e)} - \psi(e) \geq \bar{u} \tag{IR$'$}$$

$$f \geq l. \tag{LL$'$}$$

The Lagrange function is given by:

$$\mathscr{L} = p(e)\pi + \pi(\theta_H) - f - B(e) + \lambda\left[f + B(e) - \psi(e) - \bar{u}\right] + \mu[f - l],$$

where λ and μ are the associated Lagrange multipliers, and

$$B(e) := \frac{p(e)\psi'(e)}{p'(e)}.$$

Notice that

$$B'(e) = \frac{p(e)}{p'(e)}\left[\psi''(e) - \frac{p''(e)\psi'(e)}{p'(e)}\right] + \psi'(e).$$

The KKT conditions are given by:

$$p'(e)\pi - B'(e) + \lambda[B'(e) - \psi'(e)] = 0, \tag{23}$$

$$\lambda + \mu = 1, \tag{24}$$

$$\lambda\left[f + B(e) - \psi(e) - \bar{u}\right] = 0, \tag{25}$$

$$\mu[f - l] = 0, \tag{26}$$

$$\lambda, \mu \geq 0. \tag{27}$$

First notice that both individual rationality and limited liability constraints cannot be non-binding simultaneously, otherwise it would contradict the KKT condition (24) [i.e., 0=1]. Hence, we consider the following three cases.

CASE 1: $\lambda = 0$ and $\mu = 1$, i.e., individual rationality does not bind but limited liability does. In this case $f = l$. From the first order condition (23) we get

$$p'(e)\pi = B'(e). \tag{28}$$

Call the level of effort that solves the above equation $\underline{e}$. Since the individual rationality constraint does not bind, we have

$$\bar{u} < l + B(\underline{e}) - \psi(\underline{e}).$$

The above inequality implies that for low values of $\bar{u}$ the solutions $f = l$ and $e = \underline{e}$ are optimal. The optimal bonus is given by $\underline{b} = \psi'(\underline{e})/p'(\underline{e})$.

CASE 2: $\lambda, \mu \in (0, 1)$, i.e., both the constraints bind at the optimum. In this case also $f = l$. Call the solution $\hat{e}(\bar{u})$ which is implicitly defined by

$$l + B(\hat{e}(\bar{u})) - \psi(\hat{e}(\bar{u})) = \bar{u}.$$

It is easy to show that the solutions $f = l$ and $e = \hat{e}(\bar{u})$ are optimal only if

$$l + B(\underline{e}) - \psi(\underline{e}) \leq \bar{u} \leq l + B(e^*) - \psi(e^*),$$

where e^* is given by the equation $p'(e^*)\pi = \psi'(e^*)$.

CASE 3: $\lambda = 1$ and $\mu = 0$, i.e., individual rationality binds but limited liability does not. From the first order condition (23) we get

$$p'(e)\pi = \psi'(e), \tag{29}$$

which gives $e = e^*$. From the incentive compatibility constraint we have $b^* = \pi$. The optimal fixed salary is determined from the individual rationality constraint which is given by:

$$f^* = \bar{u} - [B(e^*) - \psi(e^*)].$$

The non-binding limited liability constraint implies that

$$\bar{u} \geq l + B(e^*) - \psi(e^*),$$

which is the necessary condition for the solutions $e = e^*$, $b = \pi$ and $f = f^*$ to be optimal.

Exercise 1 Solve for the optimal contracts (e, f, b) when the firm faces a linear demand $P(q) = a - q$, and $p(e) = e$ and $\psi(e) = 1/2ce^2$. Define by $\phi(\bar{u})$ the value function of the principal's maximization problem. Show that $\phi(\bar{u})$ is weakly concave in $\bar{u}$. ■

7.2 Optimal debt contracts

Debt is a prevalent contracting schemes in financial contracting between entrepreneurs and outside investors. Innes (1990) shows that a 'risky' debt contract emerges as an optimal contract under two-sided limited liability when the repayment to investors is constrained to non-decreasing in the firm's performance. Consider a penny-less risk neutral entrepreneur of a startup firm who requires to raise I, which is the setup cost, from external sources. The profit of the firm $q \in [0, \infty)$ has a distribution function $F(q \mid e)$ with the corresponding density function $f(q \mid e)$ satisfies (MLRP). Let $r(q)$ denote the performance-based repayment scheme to the investor. A risky debt contract is defined as

$$r_D(q) = \min\{q, D\}, \tag{30}$$

where D is the face value of debt. We impose the following two constraints on the repayment scheme $r(q)$:

1. Two-sided limited liability: $0 \leq r(q) \leq q$ for all $q \in [0, \infty)$;
2. monotonicity: $r'(q) \geq 0$ for all $q \in [0, \infty)$.

The two-sided limited liability constraint implies that neither the investor nor the entrepreneur will get negative incomes at any state of the nature. To rationalize the monotonicity constraint, suppose over a subset of the range of values of q that $r'(q) < 0$. Then the entrepreneur will benefit by secretly borrowing money at par from some other source, and show a higher output than the realized one.

The maximization problem of the entrepreneur is given by:

$$\max_{\{e, r(q)\}} \int_0^\infty [q - r(q)] f(q \mid e) dq - \psi(e) \tag{M$_5$}$$

$$\text{subject to} \quad \int_0^\infty r(q) f(q \mid e) dq - I = 0, \tag{IRP}$$

$$\int_0^\infty [q - r(q)] f_e(q \mid e) dq = \psi'(e), \tag{IC}$$

$$0 \leq r(q) \leq q, \tag{LL}$$

$$r'(q) \geq 0. \tag{Mon}$$

Let λ and μ be the multipliers associated with (IRP) and (IC), respectively. Then the Lagrange expression is linear in $r(q)$ with the corresponding coefficient

$$\lambda - 1 + \mu \frac{f_e(q \mid e)}{f(q \mid e)}.$$

Therefore, without invoking the monotonicity constraint, the optimal solution to $r(q)$ is given by:

$$r^*(q) = \begin{cases} q & \text{if } \frac{\lambda - 1}{\mu} > \frac{f_e(q|e)}{f(q|e)}, \\ 0 & \text{if } \frac{\lambda - 1}{\mu} \leq \frac{f_e(q|e)}{f(q|e)}. \end{cases}$$

Now (MLRP) implies that there exists an output level $\hat{q}$ such that

$$r^*(q) = \begin{cases} q & \text{if } q < \hat{q}, \\ 0 & \text{if } q \geq \hat{q}. \end{cases}$$

The above contract is certainly not monotone, and $r^*(q) \neq r_D(q)$. However, invoking (Mon), it is easy to see that the constrained optimal contract takes the form of a risky debt contract $r_D(q)$ where D is the lowest value that solves the (IRP) constraint:

$$\int_0^D q f(q \mid e) dq + [1 - F(D \mid e)] D = I.$$

Notice that the above proves that $r_D(q)$ is an optimal contract that solves the maximization problem (M$_5$) subject to the constraints (IRP)-(Mon). There might be other monotone non-debt contract that solves the entrepreneur's maximization problem. Innes (1990) shows that if there is an optimal debt contract that at which the investor obtains the same expected income compared with an optimal non-debt contract $r_{ND}(q)$, then the effort level induced by the debt contract will be strictly higher than that induced by the equivalent non-debt contract, and the entrepreneur will choose the debt contract. The proof is lengthy and technical, and hence is left as an exercise. Let us give the intuition behind the result.

Let $w_{ND}(q) = q - r_{ND}(q)$ be the income of the entrepreneur at a monotone non-debt contract. Notice that $r'_{ND}(q) > 0$ and $0 \leq r_{ND}(q) \leq q$ together imply that $r'_{ND}(q) < 1$, and hence $0 \leq w'_{ND}(q) < 1$. Let $w_D(q)$ be the income schedule of the entrepreneur at the debt contract $r_D(q)$. Clearly,

$$w_D(q) = \begin{cases} 0 & \text{if } q \leq D, \\ q - D & \text{if } q > 0. \end{cases}$$

Since $w'_D(q) = 1$ for all $q > D$, there exists a unique output level $q^* > D$ such that $w_{ND}(q) > (<)\, w_D(q)$ for $q < (>)\, q^*$. Since (MLRP) implies first order stochastic dominance, a marginal increase in effort shifts probability mass to the hight performance states. Therefore, the entrepreneur's compensation is also shifted to higher outcomes, and he has stronger incentives to exert effort. In particular, beyond q^*, the debt contract gives full marginal incentives to the entrepreneur, i.e., $w'_D(q) = 1$, while the non-debt contract gives less than full marginal incentives [$w'_{ND}(q) < 1$]. Thus for the same expected income for the investor, the debt contract induces the entrepreneur to spend more effort than any monotone non-debt contract.

8 Grossman and Hart's approach to the principal-agent problem

Grossman and Hart (1983) take an alternative approach to solve the standard principal-agent problem under moral hazard. Suppose there are only N states: $q_i \in \{q_1, \ldots, q_N\}$ with $0 \leq q_1 < \ldots < q_N$. Let $p_i(e)$ denote the probability of outcome q_i given effort choice e. Assume that the principal is risk neutral with utility function $V(q - w) = q - w$, and the agent's utility function is given by:

$$U(w, e) = \phi(e)u(w(q)) - \psi(e).$$

The above general representation contains as special cases the multiplicatively separable utility function [$\psi(e) = 0$ for all e] and the additively separable utility function [$\phi(e) = 1$ for all e]. Moreover, assume that $u(w(q))$ is continuous, strictly increasing, and concave on $(w_{min}, +\infty)$, and that $\lim_{w \to w_{min}} u(w) = -\infty$. The principal's objective is to solve

$$\max_{e, \{w_i\}} \sum_{i=1}^{N} p_i(e)(q_i - w_i) \tag{M$_6$}$$

$$\text{subject to} \quad \sum_{i=1}^{N} p_i(e)\{\phi(e)u(w_i) - \psi(e)\} \geq \bar{u}, \tag{IR}$$

$$\sum_{i=1}^{N} p_i(e)\{\phi(e)u(w_i) - \psi(e)\} \geq \sum_{i=1}^{N} p_i(\hat{e})\{\phi(\hat{e})u(w_i) - \psi(\hat{e})\} \quad \text{for all } \hat{e} \in A, \tag{IC}$$

where $w_i := w(q_i)$. We concentrate on the second best contract which is determined in the following two stages.

8.1 Implementation

This stage involves solving the following problem:

$$\min_{\{w_i\}} \sum_{i=1}^{N} p_i(e) w_i \tag{Min}$$

$$\text{subject to} \quad \sum_{i=1}^{N} p_i(e)\{\phi(e)u(w_i) - \psi(e)\} \geq \bar{u}, \tag{IR$'$}$$

$$\sum_{i=1}^{N} p_i(e)\{\phi(e)u(w_i) - \psi(e)\} \geq \sum_{i=1}^{N} p_i(\hat{e})\{\phi(\hat{e})u(w_i) - \psi(\hat{e})\} \quad \text{for all } \hat{e} \in A. \tag{IC$'$}$$

This program solves for any effort $e \in A$. Now define $h := u^{-1}$ and $u_i := u(w_i)$ for $i = 1, \ldots, N$. Let

$$\boldsymbol{U} = \{u \mid u(w) = u \text{ for some } w \in (w_{min}, +\infty)\},$$

assume that

$$\frac{\bar{u} + \psi(e)}{\phi(e)} \in \boldsymbol{U} \quad \text{for all } e \in A,$$

i.e., for every effort e, there exists a compensation level that meets the individual rationality constraint of the agent for that effort. Then the transformed program is

$$\min_{\{u_i\}} \sum_{i=1}^{N} p_i(e) h(u_i) \tag{Min$'$}$$

$$\text{subject to} \quad \sum_{i=1}^{N} p_i(e)\{\phi(e)u_i - \psi(e)\} \geq \bar{u}, \tag{IR$''$}$$

$$\sum_{i=1}^{N} p_i(e)\{\phi(e)u_i - \psi(e)\} \geq \sum_{i=1}^{N} p_i(\hat{e})\{\phi(\hat{e})u_i - \psi(\hat{e})\} \quad \text{for all } \hat{e} \in A. \tag{IC$''$}$$

We now have linear constraints and a convex objective function, and hence the KKT conditions are necessary and sufficient. Let $\boldsymbol{u} = (u_1, \ldots, u_N)$ and define

$$C(e) = \inf\left\{\sum_{i=1}^{N} p_i(e)h(u_i) \mid \boldsymbol{u} \text{ implements } e\right\}.$$

If there is no $\boldsymbol{u}$ that implements some $e \in A$, for such effort let $C(e) = +\infty$. Grossman and Hart (1983) show that when $p_i(e) > 0$ for $i = 1, \ldots, N$, there exists a solution $\boldsymbol{u}^* = (u_1^*, \ldots, u_N^*)$ to the problem (Min$'$), so that the cost function $C(a)$ is well defined. If one assumes that all relevant $h(u_i)$ are bounded and that the constraint set is compact, then $\boldsymbol{u}^*$ exists by Weierstrass' theorem.

8.2 Optimization

The second stage is to choose $e \in A$ to solve

$$\max_{e} \sum_{i=1}^{N} p_i(e) q_i - C(e).$$

To summarize, first find the minimum cost for the principal to implement a given effort level e. Once the cost function is determined, find the effort level that maximizes the principal's expected net profit.

9 Optimal contracts with multiple tasks

Hölmstrom and Milgrom (1991) analyze the optimal incentive schemes when the agent may undertake more than one tasks. The agent undertakes two tasks $i = 1, 2$ whose efforts are e_1 and e_2. The output of task i is given by $q_i = e_i + \varepsilon_i$ where ε_i is a task-specific noise which is normally distributed with mean 0 and variance σ_i^2, and is uncorrelated with ε_j where $j \neq i$. The effort-cost function is given by:

$$\psi(e_1, e_2) = \frac{1}{2}\left(c_1 e_1^2 + c_2 e_2^2\right) + \delta e_1 e_2 \quad \text{with} \quad 0 \leq \delta \leq \sqrt{c_1 c_2}.$$

When $\delta = 0$, the two efforts are technologically independent, and they are perfect substitutes if $\delta = \sqrt{c_1 c_2}$. Whenever $\delta > 0$, raising effort on one task raises the marginal cost of effort on the other task, the so-called *effort substitution* problem. The agent's risk preferences are given by a CARA utility function of the following form:

$$u(w, e_1, e_2) = -\exp\{-\eta[w - \psi(e_1, e_2)]\},$$

where η is the coefficient of absolute risk aversion, and w is a linear compensation scheme which is given by:

$$w = f + b_1 q_1 + b_2 q_2.$$

The agent's certainty equivalent compensation is given by:[1]

$$\hat{w}(e_1, e_2) = f + b_1 e_1 + b_2 e_2 - \frac{\eta}{2}\left[b_1 \sigma_1^2 + b_2 \sigma_2^2\right] - \frac{1}{2}\left(c_1 e_1^2 + c_2 e_2^2\right) - \delta e_1 e_2. \tag{31}$$

The incentive compatibility constraints are given by:

$$b_i = c_i e_i + \delta e_j \quad \text{for} \quad i, j = 1, 2, \quad \text{and} \quad i \neq j. \tag{IC$_i$}$$

Solving the above equations we get

$$e_i = \frac{b_i c_j - \delta b_j}{c_i c_j - \delta^2} \quad \text{for} \quad i, j = 1, 2, \quad \text{and} \quad i \neq j.$$

[1] For derivation see Section 6.

The principal's maximization problem is given by:

$$\max_{e_1,e_2,f,b_1,b_2} \quad e_1(1-b_1)+e_2(1-b_2)-f \tag{M$_7$}$$

$$\text{subject to} \quad f+b_1e_1+b_2e_2-\frac{\eta}{2}\left[b_1\sigma_1^2+b_2\sigma_2^2\right]-\frac{1}{2}\left(c_1e_1^2+c_2e_2^2\right)-\delta e_1e_2 \geq \bar{w}, \tag{IR}$$

$$e_1=\frac{b_1c_2-\delta b_2}{c_1c_2-\delta^2}, \tag{IC$_1$}$$

$$e_2=\frac{b_2c_1-\delta b_1}{c_1c_2-\delta^2}. \tag{IC$_2$}$$

Substitute for e_1 and e_2 from (IC$_1$) and (IC$_2$) into the objective function and the constraint (IR). So the maximization program is expressed only in terms of f, b_1 and b_2. Since the individual rationality constraint will bind at the optimum, we get $f=f(b_1,b_2)$ which we substitute in the objective function to get an unconstrained maximization problem in terms of b_1 and b_2 alone. Solving the unconstrained maximization problem we get the optimal piece rates which are given by:

$$b_1^*=\frac{1+(c_2-\delta)\eta\sigma_2^2}{1+\eta c_1\sigma_1^2+\eta c_2\sigma_2^2+\eta^2\sigma_1^2\sigma_2^2(c_1c_2-\delta^2)},$$
$$b_2^*=\frac{1+(c_1-\delta)\eta\sigma_1^2}{1+\eta c_1\sigma_1^2+\eta c_2\sigma_2^2+\eta^2\sigma_1^2\sigma_2^2(c_1c_2-\delta^2)}.$$

- When the two tasks are technologically independent, i.e., $\delta=0$, the optimal piece rates are given by:

  $$b_i^*=\frac{1}{1+\eta c_i\sigma_i^2} \quad \text{for} \quad i=1,2,$$

  the results we have obtained in Section 6.

- It is easy to check that both $\partial b_i^*/\partial\sigma_i^2$ are $\partial b_i^*/\partial\sigma_j^2$ are negative. The first one is the simple trade-off between risk and incentive. The second one is the complementarity between the piece rates in the presence of effort substitution problem.

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