CHAPTER 2: Real Analysis

1 Metric Space

Definition 1 (Metric space) *A metric space is a pair* (X, d)*, where* $X \neq \emptyset$ *and* $d : X \times X \longrightarrow \mathbb{R}$ *is a distance function or metric over* X *that associates with each pair of points* (x, y) *in* X *a real number* $d(x, y)$ *that satisfies*

(a) $d(x, y) \geq 0$ *for all* $x, y \in X$*,*

(b) $d(x, y) = 0 \Longleftrightarrow x = y$ *for all* $x, y \in X$*,*

(c) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*,*

(d) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$*. (Triangular inequality)*

Example 1 Following are the examples of metric spaces.

(a) $(\mathbb{R}, d)$ where $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$.

(b) $(\mathbb{R}^n, d_2)$ where $\mathbb{R}^n = \{(x_1, \ldots, x_n) | x_i \in \mathbb{R}$ for $i = 1, \ldots, n\}$ is the n-dimensional Euclidean space, and $d_2(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ for $x, y \in \mathbb{R}^n$.

Proof The proofs of 1(a)-(c) are trivial. Thus, we will only prove the triangular inequality which makes use of the following inequality:

$$\frac{(a_1 + \ldots + a_n)^2}{x_1 + \ldots + x_n} \leq \frac{a_1^2}{x_1} + \ldots + \frac{a_n^2}{x_n} \quad \text{for } a_i \in \mathbb{R} \text{ and } x_i \in \mathbb{R}_{++}. \tag{1}$$

To prove the above inequality, notice that, for real numbers a and b, and for $x > 0$ and $y > 0$, the following is true.

$$\frac{(a+b)^2}{x+y} \leq \frac{a^2}{x} + \frac{b^2}{y}. \tag{2}$$

The above is nothing but the restatement of $(ay - bx)^2 \geq 0$. Now replace b by $b + c$ and y by $y + z$ in (2) to get the following:

$$\frac{(a+b+c)^2}{x+y+z} \leq \frac{a^2}{x} + \frac{(b+c)^2}{y+z} \leq \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z}. \tag{3}$$

Repeat the above steps to get the inequality in (1). Take $a_i = (x_i - y_i)(y_i - z_i)$ and $x_i = (y_i - z_i)^2$. Then inequality (1) implies that

$$\sum_{i=1}^{n}(x_i - y_i)(y_i - z_i) \leq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}\sqrt{\sum_{i=1}^{n}(y_i - z_i)^2}. \tag{4}$$

The above inequality is known as the *Cauchy-Schwartz inequality*. Now we are ready to show the triangular inequality.

$$\begin{aligned}
[d_2(x,z)]^2 &= \sum_{i=1}^{n}(x_i - z_i)^2 \\
&= \sum_{i=1}^{n}\{(x_i - y_i) + (y_i - z_i)\}^2 \\
&= \sum_{i=1}^{n}(x_i - y_i)^2 + \sum_{i=1}^{n}(y_i - z_i)^2 + 2\sum_{i=1}^{n}(x_i - y_i)(y_i - z_i) \\
&\leq [d_2(x,y)]^2 + [d_2(y,z)]^2 + 2\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}\sqrt{\sum_{i=1}^{n}(y_i - z_i)^2} \\
&= [d_2(x,y)]^2 + [d_2(y,z)]^2 + 2d_2(x,y)d_2(y,z) \\
&= [d_2(x,y) + d_2(y,z)]^2.
\end{aligned}$$

The above inequality follows from (4). ■

(c) $(\mathbb{R}^n, d_\infty)$ where $d_\infty(x,y) = \max_{i=1,\ldots,n}\{|x_i - y_i|\}$. For proving the triangular inequality, notice that for all $i = 1,\ldots,n$, $\alpha_i \leq \max_j \alpha_j$ and $\beta_i \leq \max_j \beta_j$, which together imply $\alpha_i + \beta_i \leq \max_j \alpha_j + \max_j \beta_j$, which in turn implies $\max_i(\alpha_i + \beta_i) \leq \max_j \alpha_j + \max_j \beta_j$. Now take $\alpha_i = |x_i - y_i|$ and $\beta_i = |y_i - z_i|$ to complete the proof.

(d) $(\mathbb{R}^n, d_p)$ where $d_p(x,y) = \left[\sum_{i=1}^{n}|x_i - y_i|^p\right]^{\frac{1}{p}}$.

(e) $(\mathscr{C}([a,b]), d_2)$ where $\mathscr{C}([a,b]) = \{f : [a,b] \longrightarrow \mathbb{R} | f \text{ is continuous}\}$, the set of all real valued continuous functions defined on $[a,b] \subset \mathbb{R}$, and $d_2(f,g) = \left(\int [f(x) - g(x)]^2 dx\right)^{\frac{1}{2}}$.

The proofs of (c)-(e) are left as exercise.

Definition 2 (Euclidean norm) *The Euclidean norm of a vector* $x \in \mathbb{R}^n$*, denoted* $\|x\|$*, is defined as*

$$\|x\| := \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}.$$

The Euclidean norm satisfies the following properties at all $x, y \in \mathbb{R}^n$*, and* $\alpha \in \mathbb{R}$*.*

(a) Positivity: $\|x\| \geq 0$, *with equality if and only if* $x = 0$.

(b) Homogeneity: $\|\alpha x\| = |\alpha| . \|x\|$.

(c) Triangular Inequality: $\|x + y\| \leq \|x\| + \|y\|$.

It is easy to show that, for $d(x,y) := \|x-y\|$, $(\mathbb{R}^n, \|.\|)$ is a metric space.

Definition 3 (Open and closed balls) *Let (X,d) is a metric space. For $x \in X$ and $r > 0$, the* **open ball** *with centre x and radius r is given by*

$$B_r(x) = \{y \in X | d(x,y) < r\}, \tag{5}$$

and the **closed ball** *with centre x and radius r is given by*

$$\bar{B}_r(x) = \{y \in X | d(x,y) \leq r\}. \tag{6}$$

Definition 4 (Bounded set) *A subset S of X is* **bounded** *if there exist $x \in X$ and $r > 0$ such that $S \subset B_r(x)$.*

Lemma 1 *Let (X,d) be a metric space and $S \subset X$. The diameter of S is defined by*

$$diam(S) := sup\{d(x,y) | x, y \in S\}$$

Then S is bounded if and only if $diam(S)$ is finite.

Proof *Necessity:* Let $x, y \in S$. Since S is bounded, there exist $x_0 \in X$ and $r \in (0,\infty)$ such that $S \subset B_r(x_0)$. Since d is a metric, by the triangular inequality, we have

$$\begin{aligned} & d(x,y) \leq d(x,x_0) + d(x_0,y) < 2r, \\ \Longrightarrow \quad & diam(S) = sup\{d(x,y)\} < 2r < \infty. \end{aligned}$$

Sufficiency: Assume $S \neq \emptyset$ and take $\bar{x} \in S$. Consider any other $x \in S$. Then we have $d(x,\bar{x}) \leq diam(S)$. Take $r > 0$ such that $diam(S) < r$. This implies that $x \in B_r(\bar{x})$. Thus, $S \subset B_r(\bar{x})$ implying that S is bounded. ■

Definition 5 (Bounded function) *Let A be an arbitrary set, and let (X,d) be a metric space. The function $f : A \longrightarrow X$ is bounded if and only if $f(A) = \{f(a) | a \in A\} \subseteq X$ is bounded.*

2 Sequence in a Metric Space

Definition 6 (Sequence) *Let (X,d) be a metric space. A sequence in (X,d) is a function from $\mathbb{N}$, the set of natural numbers, to X that associates with each $n \in \mathbb{N}$ an element x_n of X, and is denoted by $\{x_n\}$.*

Definition 7 (Convergent sequence) *A sequence $\{x_n\}$ in a metric space (X,d) converges to a point $x \in X$ if and only if for any $\varepsilon > 0$, there exists n_ε such that $d(x_n, x) < \varepsilon$ for every $n \geq n_\varepsilon$. If a sequence $\{x_n\}$ in a metric space (X,d) converges to a point $x \in X$, then we say that the sequence is convergent and its limit is x, and write*

$$\lim_{n\to\infty} x_n = x.$$

Lemma 2 (Uniqueness of limit) *If $\{x_n\}$ converges to x and x' in a metric space (X,d), then $x = x'$.*

Proof Suppose that $x \neq x'$, and define by $r = d(x, x') > 0$. Take ε such that $0 < \varepsilon < r/2$. Since $\{x_n\}$ converges to x and x', we can find n_ε and n'_ε such that $d(x_n, x) < \varepsilon$ for $n \geq n_\varepsilon$ and $d(x_n, x') < \varepsilon$ for $n \geq n'_\varepsilon$, respectively. Take any $n \geq \max\{n_\varepsilon, n'_\varepsilon\}$. By the triangular inequality,

$$d(x, x') \leq d(x, x_n) + d(x_n, x') < 2\varepsilon < r = d(x, x'),$$

which is a contradiction. ■

Proposition 1 *We have* $\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof Let $\{x_n\}$ be a sequence such that $x_n = \frac{1}{n}$. We have to find an n_ε such that $|1/n - 0| = 1/n < \varepsilon$ for every $n \geq n_\varepsilon$. Notice that if $n \geq n_\varepsilon$, then $1/n \leq 1/n_\varepsilon$. Thus, if we pick $n_\varepsilon > 1/\varepsilon$, then $1/n_\varepsilon < \varepsilon$, and so $|1/n - 0| < \varepsilon$. Since ε can be chosen arbitrarily, $1/n$ converges to 0. ■

Lemma 3 *If* $\{x_n\}$ *is convergent, then* $\{x_n\}$ *is bounded.*

Proof Take $\varepsilon = 1$, and suppose that $\{x_n\}$ converges to x. Then there exists n_1 such that $d(x_n, x) < 1$ for all $n \geq n_1$. Now take r such that

$$r > \max\{1, d(x_1, x), \ldots, d(x_{n_1}, x)\}.$$

We have to check that $x_n \in B_r(x)$ for all $n \in \mathbb{N}$. If $n \geq n_1$, then $d(x_n, x) < 1 < r$, and hence $x_n \in B_r(x)$. If $n < n_1$, then $d(x_n, x) < r$, and hence $x_n \in B_r(x)$. ■

Let $\{x_n\} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, \ldots\}$ be a sequence. Then $\{x_2, x_4, x_6, x_8, \ldots\}$ is a subsequence of $\{x_n\}$. Formally,

Definition 8 (Subsequence) *Let* $s : \mathbb{N} \longrightarrow X$ *be a sequence and let* $p : \mathbb{N} \longrightarrow \mathbb{N}$ *that associates with each* $k \in \mathbb{N}$ *a natural number* $n_k \in \mathbb{N}$ *be a strictly increasing function, i.e.,* $p(n+1) > p(n)$ *for all* $n \in \mathbb{N}$. *Then* $s \circ p : \mathbb{N} \longrightarrow X$ *is a subsequence of* s.

Lemma 4 *If* $\{x_{n_k}\}$ *is a subsequence of* $\{x_n\}$, *then* $\lim_{n\longrightarrow\infty} x_n = x$ *implies that* $\lim_{n_k\to\infty} x_{n_k} = x$.

Proof If $p : \mathbb{N} \longrightarrow \mathbb{N}$ is strictly increasing, then $p(n) \geq n$ for all $n \in \mathbb{N}$. Fix an $\varepsilon > 0$. Then there exists $n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq n_\varepsilon$. Take $k \geq n_\varepsilon$. Since $n_k \geq k$, we have $n_k \geq n_\varepsilon$. Thus, $d(x_{n_k}, x) < \varepsilon$ for $n_k \geq n_\varepsilon$. ■

The above lemma further asserts that if two subsequences of a sequence $\{x_n\}$ converge to two different limit points, then the sequence has no limit. Thus, the converse of Lemma 3 is not always true. For example, the sequence $\{x_n\}$ such that $x_n = (-1)^n$ is bounded by -1 and 1. But the subsequence $\{x_{2n}\}$ converges to 1 and the subsequence $\{x_{2n+1}\}$ converges to -1.

Lemma 5 *Properties of a sequence in* $(\mathbb{R}, d)$ *where* $d(x, y) := |x - y|$.

(a) *A sequence* $\{x_n\}$ *that is increasing (decreasing), i.e.,* $x_n \leq (\geq) x_{n+1}$ *for all* $n \in \mathbb{N}$, *and bounded is convergent to its supremum (infimum), i.e.,* $\lim_{n\to\infty} x_n = \sup_n\{x_n\}(\inf_n\{x_n\})$.

(b) *Every sequence in* $\mathbb{R}$ *contains a monotone subsequence.*

(c) *(Bolzano-Weirstrass Theorem) Every bounded sequence in* $\mathbb{R}$ *contains a convergent subsequence.*

(d) $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$.

(e) $\lim_{n\to\infty}(x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$.

(f) $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$ *for* $\lim_{n\to\infty} y_n \neq 0$.

(g) $x_n \leq y_n$ *for all* $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} x_n$ *and* $\lim_{n\to\infty} y_n$ *exist. Then* $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.

Proof (a) Suppose that $\{x_n\}$ is an increasing and bounded (from above) sequence. By the axiom of the supremum, $s = sup_n\{x_n\}$ exists. Take an $\varepsilon > 0$. Given that $s = sup_n\{x_n\}$, $s - \varepsilon$ cannot be an upper bound of x_n, and hence there exists an $N \in \mathbb{N}$ such that $s \geq x_N > s - \varepsilon$. Since the sequence is increasing, we have that $s \geq x_n > s - \varepsilon$ for all $n \geq N$. Therefore,

$$s - \varepsilon < x_n < s + \varepsilon, \text{ for all } n > N,$$

which implies that $|x_n - s| < \varepsilon$ for all $n > N$. The proof for a decreasing sequence is analogous.

(b) Let $\{x_n\}$ be an arbitrary sequence of real numbers. The term x_s is a dominant term if for all $s < n$ we have $x_s > x_n$. Let

$$S = \{s \in \mathbb{N} | x_s > x_n \text{ for all } n > s\}$$

be the set of all subindices of the dominant terms of the sequence $\{x_n\}$. There are two possibilities. First, S is infinite. Then order the dominant terms with increasing subindices, i.e., $x_{s_1} > \ldots > x_{s_k} > \ldots$ for $s_1 < \ldots < s_k < \ldots$. Therefore, the subsequence $\{x_{s_k}\}$ is a decreasing subsequence. The second possibility is that S is finite. Then there exists s_1 such that x_n is not a dominant term for each $n \geq s_1$. Since x_{s_1} is not a dominant term, there exists $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$. Given that x_{s_2} is not a dominant term, there exists $s_3 > s_2$ such that $x_{s_2} < x_{s_3}$. Continuing in this way, we can construct an increasing subsequence.

(c) Let $\{x_n\}$ be a bounded sequence of real numbers. Property (b) tells us that $\{x_n\}$ contains at least one monotone subsequence. Clearly, this subsequence must be bounded. Thus, by Property (a) this subsequence is convergent. ■

Lemma 6 *Properties of a sequence in* $(\mathbb{R}^m, d)$.

(a) *A sequence* $\{x_n\}$ *in* $\mathbb{R}^m$ *converges to a vector* $x = (x^1, \ldots, x^m)$ *if and only if each coordinate sequence* $\{x_n^i\}$ *converges to* x^i *for each* $i = 1, \ldots, m$.

(b) *A sequence* $\{x_n\}$ *in* $\mathbb{R}^m$ *is bounded if and only if each coordinate sequence* $\{x_n^i\}$ *is bounded for each* $i = 1, \ldots, m$.

(c) *(Bolzano-Weirstrass Theorem) Every bounded sequence in* $\mathbb{R}^m$ *contains a convergent subsequence.*

3 Open and Closed Sets

Definition 9 (Open and closed sets) *Let* (X, d) *be a metric space. The set* A *in* X *is open if and only if for every* $x \in A$, *there exists* $\varepsilon > 0$ *such that* $B_\varepsilon(x) \subseteq A$. *The set* A *is closed if and only if* $X \setminus A$ *is open.*

Example 2 Following are examples of open sets.

(a) $\varnothing$ and X are open sets in X.

(b) Any open interval (a, b) of $\mathbb{R}$ with the usual metric $d(x, y) = |x - y|$ is an open set.

(c) The subset $\{(x, y) \in \mathbb{R}^2 | x > 1, y < 1\}$ of $\mathbb{R}^2$ with the metric d_2 is an open set.

(d) An open ball in $\mathbb{R}^n$ is an open set. Take an open ball $B_r(x)$ for $x \in X$, and $y \in B_r(x)$. by the definition of an open ball, $r - d(x, y) > 0$. Take $s = \frac{1}{2}[r - d(x, y)]$ and $z \in B_s(y)$. Then we have

$$\begin{aligned} & d(y, z) < s = \frac{1}{2}[r - d(x, y)], \\ \Rightarrow\ & 2d(y, z) + d(x, y) < r, \\ \Rightarrow\ & d(y, z) + d(x, y) < r, \\ \Rightarrow\ & d(x, z) \leq d(y, z) + d(x, y) < r. \end{aligned}$$

Hence, $B_s(y) \subseteq B_r(x)$. □

Example 3 Following are the examples of closed sets.

(a) $\varnothing$ and X are closed sets in X.

(b) Any closed interval $[a, b]$ of $\mathbb{R}$ with the usual metric $d(x, y) = |x - y|$ is a closed set.

(c) The subset $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq r\}$ of $\mathbb{R}^2$ with the metric d_2 is a closed set.

(d) A closed ball in $\mathbb{R}^n$ is a closed set.

□

Lemma 7 *Properties of open (closed) sets.*

(a) If $\{A_i\}_{i \in I}$ is an arbitrary collection of open (closed) sets, then $\cup_{i \in I} A_i$ is open ($\cap_{i \in I} A_i$ is closed).

(b) If $\{A_i\}_{i \in N}$ for $N = \{1, \ldots, n\}$ is a finite collection of open (closed) sets, then $\cap_{i \in N} A_i$ is open ($\cup_{i \in N} A_i$ is closed).

Proof (a) If $x \in \cup_{i \in I} A_i$, then x belongs to some particular A_i. Since A_i is open, there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A_i$, which implies that $B_\varepsilon(x) \subseteq \cup_{i \in I} A_i$.

(b) If $x \in \cap_{i \in N} A_i$, then x belongs to each A_i. Since each A_i is open, there exists an open ball $B_{\varepsilon_i}(x) \subseteq A_i$. Notice that the smallest such ball is contained in all the A_i's, and hence in their finite intersection. That is, if we take $\varepsilon = \min_i\{\varepsilon_i\}$, then $B_\varepsilon(x) \subseteq B_{\varepsilon_i}(x) \subseteq A_i$ for each $i \in N$, which implies that $B_\varepsilon(x) \subseteq \cap_{i \in N} A_i$. Thus, $\cap_{i \in N} A_i$ is open. ■

The finite intersection property is important. Consider the following infinite family of open sets $\{(-1, 1), (-1/2, 1/2), \ldots, (-1/n, 1/n), \ldots\}$ whose intersection is $\{0\}$, which is a closed set.

Definition 10 (Interior, Exterior, Boundary and Closure) *Let (X, d) be a metric space, $A \subseteq X$ and $x \in X$.*

(a) *A point x_{int} is an interior point of A if there exists $\varepsilon > 0$ such that $B_\varepsilon(x_{int}) \subseteq A$. The set of all interior points of A is denoted by $int(A)$.*

(b) *A point x_{ext} is an exterior point of A if there exists $\varepsilon > 0$ such that $B_\varepsilon(x_{ext}) \subseteq A^c$. The set of all exterior points of A is denoted by $ext(A)$.*

(c) *A point x_{bd} is a boundary point of A if for any $\varepsilon > 0$, the open ball $B_\varepsilon(x_{bd})$ has non-empty intersection with both A and A^c, i.e., $B_\varepsilon(x_{bd}) \cap A \neq \emptyset$ and $B_\varepsilon(x_{bd}) \cap A^c \neq \emptyset$. The set of all boundary points of A is denoted by $bd(A)$.*

(d) *A point x_{cl} is a closure point of A if for any $\varepsilon > 0$, the open ball $B_\varepsilon(x_{cl})$ has a non-empty intersection with A, i.e., $B_\varepsilon(x_{cl}) \cap A \neq \emptyset$. The set of all closure points of A is denoted by $cl(A)$.*

Since any interior point of A lies inside an open ball contained in A, we have $int(A) \subseteq A$. In the same manner, $ext(A) \subseteq A^c$. Now let an $x \in A$. Any open ball around it contains the point itself, and hence $x \in cl(A)$. Thus we have $int(A) \subseteq A \subseteq cl(A)$. It is also easy to see that $int(A)$, $bd(A)$ and $ext(A)$ constitute a partition of X, i.e., $int(A) \cup bd(A) \cup ext(A) = X$. Finally, we have $cl(A) = int(A) \cup bd(A)$ and $ext(A) = int(A^c)$. Using the above concepts, we get the following characterizations of open and closed sets.

Proposition 2 *For an arbitrary set A,*

(a) *the set $int(A)$ is the largest open set contained in A,*

(b) *A is open if and only if $int(A) = A$,*

(c) *the set $cl(A)$ is the smallest closed set that contains A,*

(d) *A is closed if and only if $cl(A) = A$.*

Proof (a) We have shown that $int(A) \subseteq A$. Next take $x \in int(A)$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$. Now, following Example 2(d) show that $y \in B_\varepsilon(x)$ implies that $y \in int(A)$. Thus, $B_\varepsilon(x) \subseteq int(A)$, and hence $int(A)$ is open. Finally, take a set B that is an open subset of A. We need to show that $B \subseteq int(A)$. Consider an $x \in B$. Since B is open, for an $\varepsilon > 0$ we have that $B_\varepsilon(x) \subseteq B \subseteq A$. Therefore, $x \in int(A)$ which implies that $B \subseteq int(A)$.

(b) If $int(A) = A$, then A is open because $int(A)$ is open. If A is open, then its largest open subset is A itself, and hence $int(A) = A$. ■

Example 4 (Efficient production) Let $y = (y_1, y_2)$ be a production plan where $y_1 \in \mathbb{R}_-$ represents an input and $y_2 \in \mathbb{R}_+$ represents an output. For example, $y = (-2, 3)$ implies that 2 units of an input produce 3 units of output. The set $Y = \{y \in \mathbb{R}^2 \mid y \text{ is a production plan}\}$ is called the production possibility set, which is the set of all feasible production plans. A production plan y is efficient if and only if there is no $y' \in Y$ with $y' \geq y$, i.e., it is not possible to produce the same output with less input or more output with same input. Let $Eff(Y) = \{y \in Y \mid y' \geq y \Rightarrow y' \notin Y\}$ be the set of efficient productions of Y. Notice that every interior point of Y is an inefficient plan. Take an $y^0 \in int(Y)$. Then there exists an ε-ball $B_\varepsilon(y^0)$ around y^0 which is contained in Y. This ball contains at least a plan $y' \geq y^0$. Therefore, $y^0 \notin Eff(Y)$. Hence if $y \in Eff(Y)$, then $y \in bd(Y)$, i.e., $Eff(Y) \subseteq bd(Y)$. Convince yourselves that the converse is not true in general. □

In what follows, we provide yet another characterization of closed sets.

Definition 11 (Limit points) *Let (X, d) be a metric space and A be a set in X. A point x in X is a limit (cluster) point of A if every open ball around it contains at least one point of A, which is distinct from x. The set of all limit points of A is called its derived set and is denoted by $D(A)$. Formally, $x \in D(A)$ if and only if for each $\varepsilon > 0$ we have $B_\varepsilon(x) \cap (A \setminus \{x\}) \neq \varnothing$.*

Notice that the concept of limit points is more restrictive than that of the closure points. Define by $I(A) := cl(A) \setminus D(A)$, which is the set of all isolated points of A. In other words, a point y is in $I(A)$ if and only if there exists $\varepsilon > 0$ such that $B_\varepsilon(y) \cap A = \{y\}$. Thus, closure points of a set A are either its limit points or its isolated pointas, because $D(A) \cup I(A) = cl(A)$.

Proposition 3 *Let (X, d) be a metric space and A be a set in X. A point $x \in X$ is in $D(A)$ if and only if there exists a sequence in $A \setminus \{x\}$ that converges to x.*

Proof First suppose that there is a sequence $\{x_n\} \subset A \setminus \{x\}$ that converges to x. Then for any given $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq n_\varepsilon$, which implies that $x_n \in B_\varepsilon(x)$ for each $n \geq n_\varepsilon$. Thus we have $B_\varepsilon(x) \cap (A \setminus \{x\}) \neq \varnothing$ for this ε. Since this is true for any $\varepsilon > 0$, $x \in D(A)$.

Next suppose that $x \in D(A)$. Then for all $\varepsilon > 0$, $B_\varepsilon(x) \cap (A \setminus \{x\}) \neq \varnothing$. We will construct a sequence with the desired property. Take $\varepsilon = 1$. Since $B_1(x) \cap (A \setminus \{x\}) \neq \varnothing$, there is some $x_1 \in B_1(x) \cap (A \setminus \{x\})$. Next take $\varepsilon = 1/2$. Since $B_{1/2}(x) \cap (A \setminus \{x\}) \neq \varnothing$, there is some $x_2 \in B_{1/2}(x) \cap (A \setminus \{x\})$. Continuing this way, we can have some $x_n \in B_{1/n}(x) \cap (A \setminus \{x\})$. Thus, we have constructed a sequence $\{x_n\} \subset A \setminus \{x\}$ with the property that

$$0 \leq d(x_n, x) < \frac{1}{n}.$$

As $n \to \infty$, $\frac{1}{n} \to 0$ and $d(x_n, x) \to 0$. Therefore, $\{x_n\}$ converges to x. ■

Theorem 1 *Let (X, d) be a metric space and A be a set in X. Then the following three statements are equivalent.*

(a) A is closed.

(b) A contains all its limit points, i.e., $D(A) \subseteq A$.

(c) Every convergent sequence in A has its limit in A, i.e., if $x_n \in A$ for all n and $\{x_n\}$ converges to x, then $x \in A$.

Proof First, we show the equivalence between (a) and (b). Assume that A is closed, then $X \setminus A$ is open. Then for any $x \in X \setminus A$, there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset X \setminus A$, which implies that $B_\varepsilon(x) \cap A = \varnothing$. Thus, no point in $X \setminus A$ can be a limit point of A, and hence all such points must be contained in A. To show the converse, suppose that $X \setminus A$ is not open, i.e., A is not closed. Then there is a point $x \in X \backslash A$ such that for every $\varepsilon > 0$ the open ball $B_\varepsilon(x)$ is not entirely contained in $X \backslash A$, which implies that $B_\varepsilon(x) \cap (A \setminus \{x\}) \neq \varnothing$. Thus $x \in D(A)$ which lies in $X \setminus A$, and hence A does not contain all its limit points.

Next, we show the equivalence between (a) and (c). Let A be closed. If a sequence of A has a limit then either this limit coincides with one of the elements of the sequence (and then it lies in A) or it is a limit point of A (which is contained in A since A is closed). Therefore A contains the limits of all its convergent sequences. Conversely, suppose that the limit of any convergent sequence of A lies in A. Every limit point of A is a limit of some sequence $\{x_n\}$ in A. Therefore A contains all its limit points, and hence is closed. ■

Example 5 (Continuous preferences) Let $X \subseteq \mathbb{R}^2$ be the consumption set of an indiviual. A preference relation $\succsim$ is continuous if for any $x \in X$, its upper contour set $U_x = \{y \in X | y \succsim x\}$ and its lower contour set $L_x = \{y \in X | y \precsim x\}$ are both closed. The lexicographic preferences are discontinuous. To see this, consider the following sequences of commodity vectors $x^n = (1+1/n, 0)$ and $y^n = (1, 1)$ such that $x^n \succ y^n$ for all $n \in \mathbb{N}$. But $\lim_{n\to\infty} y^n = (1, 1) \succ (1, 0) = \lim_{n\to\infty} x^n$. Thus, the ordering is reversed at the limit, and it is discontinuous. □

4 Continuous Functions

Definition 12 (Continuity) *Let $f : S \longrightarrow T$, where $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$. Then f is said to be continuous at $x \in S$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $y \in S$ and $d(x, y) < \delta$ implies that $d(f(x), f(y)) < \varepsilon$. A function $f : S \longrightarrow T$ is said to be continuous on S if it is continuous at each point in S.*

A function f is continuous at x if the value of f at any point y that is close to x is a good approximation of the value of f at x. Thus, the identity function $f(x) = x$ for all $x \in \mathbb{R}$ is continuous at each $x \in \mathbb{R}$, while the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x \leq 0, \end{cases}$$

is continuous everywhere except at $x = 0$. At $x = 0$, every open ball $B_\delta(x)$ contains at least one $y > 0$. At all such points, $f(y) = 1 > -1 = f(x)$. Continuity can also be defined in terms of sequences as follows.

Definition 13 (Continuity) *Let $f : S \longrightarrow T$, where $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$. Then f is said to be continuous at $x \in S$ if for all sequences $\{x_n\}$ such that $x_n \in S$ for all n, and $\lim_{n\to\infty} x_n = x$, it is the case that $\lim_{n\to\infty} f(x_n) = f(x)$.*

Lemma 8 (Composition of continuous functions) *Composition of continuous functions is continuous. Let $f : S \longrightarrow T$ and $g : T \longrightarrow U$ be two functions, where $S \subset \mathbb{R}^n$, $T \subset \mathbb{R}^m$ and $U \subset \mathbb{R}^l$. If f is continuous at $x \in S$ and g is continuous at $f(x) \in T$, then $g \circ f$ is continuous at $x \in S$.*

Theorem 2 *Let $f : S \longrightarrow T$ be a function, where $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$. The following three statements are equivalent.*

(a) *f is continuous on S.*

(b) *For every closed subset C of T, $f^{-1}(C) \subset S$ is closed.*

(c) *For every open subset O of T, $f^{-1}(O) \subset S$ is open.*

Proof First, we show that (a) implies (b). The inverse image of C is given by

$$f^{-1}(C) = \{x \in S | f(x) \in C \subset T\}.$$

Let C be an arbitrary closed subset of T and x be an arbitrary limit point of $f^{-1}(C)$. Then there exists a sequence $\{x_n\}$ in $f^{-1}(C)$ that converges to x. By continuity of f, $\{f(x_n)\}$ converges to $f(x)$. By construction, $\{f(x_n)\}$ is a sequence in C. Since C is closed, $f(x)$ must lie in C, which implies that $x \in f^{-1}(C)$. Thus, $f^{-1}(C)$ contains all its limit points, and hence is closed.

Second, we show that (b) implies (c). Notice that, for an arbitrary subset O of T, $f^{-1}(O) = S \setminus f^{-1}(T \setminus O)$. To show this, pick an $x \in f^{-1}(O)$, which implies that $f(x) \in O \subset T$. Therefore, $f(x) \notin T \setminus O$, which implies that $x \notin f^{-1}(O)$, and hence $x \in S \setminus f^{-1}(T \setminus O)$. Thus, $f^{-1}(O) \subset S \setminus f^{-1}(T \setminus O)$. On the other hand, pick an $x \in S \setminus f^{-1}(T \setminus O)$, which implies that $x \notin f^{-1}(T \setminus O)$, and hence $f(x) \notin T \setminus O$. Therefore, $f(x) \in O$ implying that $x \in f^{-1}(O)$. Thus, $S \setminus f^{-1}(T \setminus O) \subset f^{-1}(O)$. Therefore, $f^{-1}(O) = S \setminus f^{-1}(T \setminus O)$. Since O is an open subset of T, $T \setminus O$ is closed, and by continuity of f, $f^{-1}(T \setminus O)$ is closed. Therefore, $f^{-1}(O) = S \setminus f^{-1}(T \setminus O)$ is open.

Finally, we show that (c) implies (a). Pick an $x \in S$, and suppose that f is not continuous at x but the inverse image of an arbitrary subset O of T is open. We show that these will lead us to a contradiction. Since, by assumption, f is not continuous, there exist some $\varepsilon > 0$ and some $x^\delta \in S$ such that for all $\delta > 0$, $d(f(x^\delta), f(x)) \geq \varepsilon$ for $d(x^\delta, x) < \delta$, i.e., $f(x^\delta)$ does not lie in the open ball $B_\varepsilon(f(x))$ for $d(x^\delta, x) < \delta$, which implies that $x^\delta \notin f^{-1}(B_\varepsilon(f(x)))$, and hence $x^\delta \in S \setminus f^{-1}(B_\varepsilon(f(x)))$. Take $\delta = 1/n$. Thus, $x_n = x^\delta$. Since $d(x_n, x) \in [0, 1/n)$, $d(x_n, x) \to 0$ as $n \to \infty$. Thus, $x_n \to x$. Now, by assumption, $f^{-1}(B_\varepsilon(f(x)))$ is open since $B_\varepsilon(f(x))$ is open. Then the set $S \setminus f^{-1}(B_\varepsilon(f(x)))$ is closed. Since $\{x_n\}$ is a sequence in $S \setminus f^{-1}(B_\varepsilon(f(x)))$ which is a closed set, we have

$$x \in S \setminus f^{-1}(B_\varepsilon(f(x))). \tag{7}$$

Notice that $f(x) \in B_\varepsilon(f(x))$ since an open ball contains its center. This implies

$$x \in f^{-1}(B_\varepsilon(f(x))). \tag{8}$$

Thus, (7) and (8) contradict each other. ■

The above theorem states that a function is continuous if and only if the inverse image of any closed (open) set under it is closed (open). This is true for the inverse images of a continuous function. A continuous function does not necessarily map closed (open) sets into closed (open) sets, and a function that maps closed (open) sets into closed (open) sets is not necessarily continuous. Try to find examples for this assertion.

5 Compact Metric Spaces

Definition 14 (Compactness) *Consider the metric space* $(\mathbb{R}^n, d)$. *A set* $A \subseteq \mathbb{R}^n$ *is compact if it is closed and bounded.*

Definition 15 (Sequential compactness) *Consider the metric space* $(\mathbb{R}^n, d)$. *A set* $A \subseteq \mathbb{R}^n$ *is sequentially compact if every sequence in* A *has a convergent subsequence whose limit lies in* A.

It can be shown that a subset A of $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded. Interested students should refer to Rudin (1976, Theorem 2.41).

Example 6 Following are the examples of compact sets.

(a) A finite set is compact.

(b) A closed interval $[a, b]$ of $\mathbb{R}$ is compact.

(c) The subset $[a, b]^n$ of $\mathbb{R}^n$ is compact. □

Theorem 3 *Suppose that* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *is a continuous function. If* $A \subset \mathbb{R}^n$ *is compact, then* $f(A) = \{y \in \mathbb{R} \mid y = f(x)$ *for some* $x \in A\}$ *is a compact subset of* $\mathbb{R}$.

Proof Pick any sequence $\{y_n\}$ in $f(A)$. For each n, pick $x_n \in A$ such that $f(x_n) = y_n$. This gives us a sequence $\{x_n\}$ in A. Since A is compact, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, and a point $x \in A$ such that $\{x_{n_k}\}$ converges to x. Define $y = f(x)$ and $y_{n_k} = f(x_{n_k})$. By construction, $\{y_{n_k}\}$ is a subsequence of $\{y_n\}$. Moreover, since $x \in A$, it is the case that $y = f(x) \in f(A)$. Since, f is continuous, $y_{n_k} = f(x_{n_k})$ must converge to $y = f(x)$. Thus, $f(A)$ is compact. ■

Theorem 4 (Weirstrass) *Let* $A \subset \mathbb{R}^n$ *be compact, and* $f : A \longrightarrow \mathbb{R}$ *is continuous on* A. *Then* f *attains both its maximum and minimum on* A, *i.e., there exist points* x_{max} *and* x_{min} *in* A *such that*

$$f(x_{max}) = \sup(f(A)) \quad \textit{and} \quad f(x_{min}) = \inf(f(A)).$$

Proof By the previous theorem, $f(A)$ is compact, and hence is a bounded subset of $\mathbb{R}$. Thus, $a = \sup(f(A))$ exists. Notice that, for each $n \in \mathbb{N}$, $a - 1/n$ is not an upper bound of $f(A)$. Thus we can find an $x_n \in A$ such that $a \geq f(x_n) > a - 1/n$. Then $n \to \infty$ implies that $f(x_n) \to a$. Thus a is a limit point of $f(A)$. Since $f(A)$ is closed, $a \in f(A)$. Hence, there exists $x_{max} \in A$ such that $a = f(x_{max})$. The existence of the minimum is shown in a similar fashion. ■

Notice that compactness of A and continuity of f are indispensable for Theorem 4. Consider the following two cases.

(a) Let $f(x) = \frac{1}{x}$ and $A = (0, 1]$. It is easily seen that f does not have a maximizer since A is not compact.

(b) Let $A = [0, 1]$ and f is given by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ \frac{1}{x}, & \text{otherwise.} \end{cases}$$

Then f does not have a maximizer since the function is not continuous.

6 Fixed Point Theorems

6.1 Intermediate Value Theorem

In this section we introduce an important property pertaining to the continuous functions: the intermediate value theorem that will be useful for the fixed point theorems.

Theorem 5 (Intermediate value theorem in $\mathbb{R}$) *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function, where $[a, b]$ is a compact interval of $\mathbb{R}$. If $f(a) < f(b)$, then for every real number $\gamma \in (f(a), f(b))$ there exists $c \in (a, b)$ such that $f(c) = \gamma$.*

Proof Let $S := \{x \in [a, b] \mid f(x) \leq \gamma\}$ and $c := \sup(S)$. Notice that S is non-empty since it contains at least a, and is bounded above by b. Hence, c is well defined by the supremum property. Notice also that, by construction, $f(x) > \gamma$ for all $x \in (c, b]$. We will show that $f(c) = \gamma$. Suppose first that $f(c) > \gamma$, i.e., $f(c) - \gamma > 0$. Since f is continuous, there exists $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ for every $x \in (c - \delta, c + \delta)$. Choose $\varepsilon = f(c) - \gamma > 0$. Then $f(x) > f(c) - [f(c) - \gamma] = \gamma$ for every $x \in (c - \delta, c + \delta)$. Thus, $c - \delta$ is an upperbound of S, contradicting the fact that $c = \sup(S)$. Now suppose that $f(c) < \gamma$, i.e., $\gamma - f(c) > 0$. Continuity of f, for $\varepsilon = \gamma - f(c) > 0$, implies that $f(x) < f(c) + [\gamma - f(c)] = \gamma$ for every $x \in (c - \delta, c + \delta)$. Thus, there are points x strictly greater than c for which $f(x) < \gamma$ which contradicts the definition of c. Therefore, $f(c) = \gamma$. ■

6.2 Brouwer's Fixed Point Theorem

Definition 16 (Fixed point) *Given a set X and a function $f : X \longrightarrow X$, the point x^* is said to be a fixed point of f if $x^* = f(x^*)$.*

Our objective is to look for sufficient conditions under which a function from a set into itself has a fixed point. To fix ideas, let $f : [0, 1] \longrightarrow [0, 1]$ be defined by $f(x) = x^2$. Notice that $f(0) = 0$, $f(1) = 1$, and $f(x) < x$ (i.e., the graph of f lies strictly below the diagonal of the unit square) for all $x \in (0, 1)$. Thus, the set of fixed points of f is given by $\mathscr{E}(f) = \{0, 1\}$. Next, consider $g : [0, 1] \longrightarrow [0, 1]$ be defined by $g(x) = 1/4 + (1/2)x$. The set fixed points of g is given by $\mathscr{E}(g) = \{1/2\}$. Now consider $h : [0, 1] \longrightarrow [0, 1]$ be defined as $h(x) = 3/4$ for $x \in [0, 1/2]$, and $h(x) = 1/4$ for $x \in (1/2, 1]$. It is easy to check that the set fixed points of h is given by $\mathscr{E}(h) = \varnothing$. The function h fails to have a fixed point because of its discontinuity at $x = 1/2$. The following theorem guarantees the existence of a fixed point.

Theorem 6 (Brouwer's fixed point theorem) *Let $f : [0, 1] \longrightarrow [0, 1]$ be a continuous function. Then the set of fixed points of f given by $\mathscr{E}(f) = \{x \in [0, 1] \mid x = f(x)\}$ is non-empty.*

The above theorem is a trivial consequence of the intermediate value theorem in $\mathbb{R}$. Let $f:[0,1]\longrightarrow[0,1]$ be a continuous function. Define by $g(x):=f(x)-x$. Notice that the function $g:[0,1]\longrightarrow[0,1]$ is continuous on with $g(0)\geq 0$ and $g(1)\leq 0$. If one of these two expressions holds with equality, then either 0 or 1 is a fixed point of f. Otherwise, by the intermediate value theorem, there is $x^*\in(0,1)$ such that $g(x^*)=0$ implying that $f(x^*)=x^*$, and hence x^* is a fixed point of f.