CHAPTER 1: Sets and Relations

1 Binary Relations

In this chapter we are going to define *relation* formally. A relation in everyday life shows an association of objects of a set with objects of other sets (or the same set) such as John owns a BMW, Jim has a green Audi, etc. The essence of relation is these associations. A collection of these individual associations is a relation, such as the ownership relation between people and automobiles. To represent these individual associations, a set of "related" objects, such as John and a BMW, can be used. However, simple sets such as {John, BMW} are not sufficient here. The order of the objects must also be taken into account, because John owns a BMW but the BMW does not own John, and simple sets do not deal with orders. Thus sets with an order on its members are needed to describe a relation. Here the concept of ordered pair is going to be defined first. A relation is then defined as a set of ordered pairs.

Definition 1 (Ordered pair) *An ordered pair is a list of a pair of objects with an order associated with them. If objects are represented by x and y, then we write an ordered pair as (x, y) or (y, x). In general (x, y) is different from (y, x).*

Two ordered pairs (x, y) and (x', y') are equal if and only if $x = x'$ and $y = y'$. For example, if the ordered pair (x, y) is equal to $(1, 2)$, then $x = 1$, and $y = 2$. The ordered pair $(1, 2)$ is not equal to the ordered pair $(2, 1)$.

Definition 2 (Cartesian product) *A Cartesian product of two non-empty sets X and Y, denoted as $X \times Y$, is defined as the set of all ordered pairs (x, y) where x is an element of X and y is an element of Y. That is,*

$$X \times Y := \{(x, y) | x \in X \text{ and } y \in Y\}.$$

Definition 3 (Binary relation) *Let X and Y be two non-empty sets. A subset R of $X \times Y$ is called a binary relation from X to Y. In other words, a binary relation from a set X to a set Y is a set of ordered pairs (x, y) where x is an element of X and y is an element of Y.*

Example 1 If $X = \{1, 2, 3\}$ and $Y = \{4, 5\}$, then $\{(1, 4), (2, 5), (3, 5)\}$, for example, is a binary relation from X to Y. However, $\{(1, 1)\}$ is not a binary relation from X to Y because 1 is not in Y. □

If $X = Y$, i.e., if R is a relation from X to X, we simply say that it is a relation on X. In other words, R is a relation on X if and only if $R \subseteq X^2$. If $(x, y) \in R$, then we think of R as associating the object x with the object y, and if $\{(x, y), (y, x)\} \cap R = \emptyset$, then we understand that there is no connection between x and y as envisaged by R. Conventionally, we write xRy instead of $(x, y) \in R$.

2 Preorders and Equivalence Relations

Definition 4 (Preorder) *A binary relation $\succsim$ on a non-empty set X is called a preorder if it has the following properties:*

(i) Reflexivity: $x \succsim x$ *for all* $x \in X$,

(iii) Transitivity: $x \succsim y$ *and* $y \succsim z \Longrightarrow x \succsim z$ *for all* $x, y, z \in X$.

Definition 5 (Equivalence relation) *A binary relation $\sim$ on a non-empty set X is called an equivalence relation if it has the following properties:*

(i) Reflexivity: $x \sim x$ *for all* $x \in X$,

(ii) Symmetry: $x \sim y \Longrightarrow y \sim x$ *for all* $x, y \in X$,

(iii) Transitivity: $x \sim y$ *and* $y \sim z \Longrightarrow x \sim z$ *for all* $x, y, z \in X$.

For any $x \in X$, the equivalence class of x is defined as the set

$$[x] := \{y \in X | y \sim x\}.$$

Definition 6 (Partition) *A partition of a non-empty set X is a class $\{X_i\}$ of non-empty subsets of X such that (i) $\bigcup_i X_i = X$, and (ii) $X_i \cap X_j = \varnothing$ for any two X_i and X_j in $\{X_i\}$. The X_i's are called the partition sets.*

If $X = \{1, 2, 3, 4, 5\}$, then $\{\{1, 2, 3\}, \{4, 5\}\}$ and $\{\{1, 2, 5\}, \{3, 4\}\}$ are two different partitions of X. If $X = \mathbb{R}$, then it can be partitioned into the infinitely many closed-open intervals of the form $[n, n+1)$ where n is an integer.

We now show that a given equivalence relation $\sim$ on X determines a natural partition of X. Let the relation $\sim$ on X satisfies reflexivity, symmetry and transitivity. We show that all distinct equivalence classes form a partition of X. By reflexivity, $x \in [x]$ for each element $x \in X$, so each equivalence set is non-empty and their union is X. It remains to show that any two equivalence sets $[x_1]$ and $[x_2]$ are either disjoint or identical. We prove by showing that if $[x_1]$ and $[x_2]$ are not disjoint then they must be identical. Suppose that $z \in [x_1] \cap [x_2]$. Then $z \sim x_1$ and $z \sim x_2$, and by symmetry, $x_1 \sim z$. Let $y \in [x_1]$ which implies that $y \sim x_1$. Since $y \sim x_1$ and $x_1 \sim z$, transitivity implies that $y \sim z$. Then, again by transitivity, $y \sim z$ and $z \sim x_2$ implies that $y \sim x_2$, so that $y \in [x_2]$. Hence, $[x_1] \subseteq [x_2]$. By a similar logic, one can show that $[x_2] \subseteq [x_1]$. From this we can conclude that $[x_1] = [x_2]$.

The converse statement is somewhat trivial. Consider a partitions $\{X_i\}$ of a set X, and let R be a relation on X defined by, for any two elements x and y of X,

$$xRy \text{ if and only if } x \text{ and } y \text{ belong to the same element } X_i \text{ of } \{X_i\}.$$

We think of the elements of X being distributed into a number of sets, each of which is a member of a given partition. Every element of X is exactly in one set. Notice that every element x of X is in the same set as itself. For any x and y of X, if x and y are in the same set, so are y and x. Finally, For any

x, y and z of X, if x and y are in the same set, and y and z are also, so are x and z. Thus, the relation R is reflexive, symmetric and transitive, and hence is an equivalence relation. Thus, there is no real distinction between partitions and equivalence relations.

Example 2 (The Indifference Set) Let $x = (x_1, x_2)$ be the quantities of two goods 1 (apple) and 2 (orange), and x is an element of $X \subseteq \mathbb{R}^2_+$, which is called the consumption set. Define the *preference relations* $\succsim$ on X of a consumer as follows. We read $x \succsim y$ as "x is at least as good as y". From $\succsim$, we can derive two other important relations on X:

(a) The *strict preference* relation, $\succ$, defined by $x \succ y \iff x \succsim y$ but $y \not\succsim x$ and read "x is (strictly) preferred to y".

(d) The *indifference* relation, $\sim$, defined by $x \sim y \iff x \succsim y$ and $y \succsim x$ and read "x is indifferent to y".

The relation $\succsim$ is *rational* if (i) it is *complete*, i.e., for all $x, y \in X$, we have that $x \succsim y$ or $y \succsim x$ (or both), and (ii) it is *transitive*, i.e., for all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$. It is easy to prove the following property.

Lemma 1 *If $\succsim$ is rational, then the indifference relation $\sim$ is an equivalence relation, but the strict preference relation $\succ$ is not.*

For a consumption bundle $x \in X$, the set $[x] = \{y \in X | y \sim x\}$ is called the *indifference set* of x. Notice that $\sim$ induces a partition of X and vice versa. □

3 Partial Orders

Definition 7 (Partial order) *A binary relation $\succsim$ on a non-empty set X is called a partial order if it has the following properties:*

(i) Reflexivity: $x \succsim x$ *for all* $x \in X$,

(ii) Antisymmetry: $x' \succsim x''$ *and* $x'' \succsim x'$ *imply* $x' = x''$ *for all* $x', x'' \in X$,

(iii) Transitivity: $x' \succsim x''$ *and* $x'' \succsim x'''$ *imply* $x' \succsim x'''$ *for all* x', x'' *and* $x''' \in X$.

Definition 8 (Partially ordered set) *A partially ordered set (written "poset") is a set X on which there is a binary relation $\succsim$ that is reflexive, antisymmetric and transitive.*

Definition 9 *Two elements x' and x'' of a partially ordered set are* ordered *if either $x' \succsim x''$ or $x'' \succsim x'$; otherwise x' and x'' are unordered.*

Definition 10 (Chain) *A partially ordered set is a chain if it does not contain an unordered pair of elements.*

Example 3 The following are partially ordered sets.

(a) The real line $\mathbb{R}$ with usual ordering relation $\geq$ on the real numbers is a poset.

(b) The m-dimensional Eucledian space $\mathbb{R}^m = \{x = (x_1, \ldots, x_m) \mid x_i \in \mathbb{R}$ for all $i = 1, \ldots, m\}$ with the vector ordering relation $\geq$ is a poset.

(c) The power set, $\mathscr{P}(X)$, of a set X is the set of all subsets of X. The power set $\mathscr{P}(X)$ with the set inclusion ordering relation $\supseteq$ is a poset. If X' and X'' are distinct subsets of X with $X' \subseteq X''$, then $X' \subset X''$.

(d) The lexicographic ordering relation $\succeq_{lex}$ on $\mathbb{R}^m$ is such that $x'' \succeq_{lex} x'$ in $\mathbb{R}^m$ if either $x' = x''$ or there is some i' with $1 \leq i' \leq m$, $x'_i = x''_i$ such that for each i with $1 \leq i \leq i'$, and $x'_{i'} < x''_{i'}$. The set $\mathbb{R}^m$ with the ordering relation $\succeq_{lex}$is a poset, indeed, is a chain. □

Definition 11 (Bound of a set and bounded set) *Suppose that X is a partially ordered set with respect to $\succsim$, and X' is a subset of X. If $u \in X$ such that $u \succsim x$ for every $x \in X'$, then u is an upper bound of X'. Similarly, if $l \in X$ such that $l \precsim x$ for every $x \in X'$, then l is a lower bound of X'. If a subset X' of a poset X has an upper (lower) bound, then X' is bounded above (below). In case X' has both an upper bound and a lower bound, then X' is a bounded set.*

First, observe that a subset X' of a poset X may be unbounded. Consider $X = \mathbb{R}$, the set of real numbers, and $X' = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of integers. Then $\mathbb{Z}$ is an unbounded set in $\mathbb{R}$. On the other hand, if $X' = \mathbb{N} = \{1, 2, \ldots\}$, the set of natural numbers, then X' is bounded below by 1. Next, let $X = \mathbb{R}$ and $X' = \{5, 10, 15\}$. Then both 15 and 20 are upper bounds of X'. Now let $X = \mathbb{R}$ and $X' = [a, b)$. The real number b is an upper bound of X' which is not contained in X'. Two important observations emerge. First, a subset X' of a poset X may or may not contain its upper (lower) bounds (if they exist). This gives rise to the notion of a greatest element and a least element.

Definition 12 (Greatest and least elements) *Suppose that X is a partially ordered set with respect to $\succsim$, and X' is a subset of X. If $x' \in X'$ such that $x' \succsim x$ for every $x \in X'$, then x' is a greatest element of X'. Similarly, if $z' \in X'$ such that $z' \precsim x$ for every $x \in X'$, then z' is a least element of X'.*

A greatest (least) element of X' is trivially an upper (lower) bound of X'. For example, if $X = \mathbb{R}$ and $X' = \{5, 10, 15\}$, then 2 is not a least element of X', whereas 5 is. Notice that a subset X' of a poset X can have at most one greatest (least) element. Next, a subset X' of a poset X may have many different upper and lower bounds. This gives rise to the notion of a supremum and an infimum.

Definition 13 (Supremum and infimum) *Suppose that X is a partially ordered set with respect to $\succsim$, and X' is a subset of X. An element of X is a supremum or a least upper bound of X' (with respect to X), denoted* $\sup_X(X')$, *if it is the case that* (a) $\sup_X(X') \succsim x$ *for all $x \in X'$, and* (b) *for any $u \in X$ such that $u \succsim x$ for all $x \in X'$, it holds that* $\sup_X(X') \precsim u$. *Similarly, an element of X is an infimum or a greatest lower bound of X' (with respect to X), denoted* $\inf_X(X')$, *if it is the case that* (a) $\inf_X(X') \precsim x$ *for all $x \in X'$, and* (b) *for any $l \in X$ such that $l \precsim x$ for all $x \in X'$, it holds that* $\inf_X(X') \succsim u$.

The above definition says that if the set of upper (lower) bounds of X' has a least (greatest) element, then this least (greatest) element is the supremum (infimum) of X'. Two important points must be kept in mind. First, a supremum or an infimum of a set may not exist. If it exists, then it must be unique. Second, one must be clear about the underlying set in expressing $\sup(X')$ or $\inf(X')$. Consider the following example.

Example 4 Suppose $X = \mathbb{R}$, $Y = [0,1) \cup \{2\}$, and $X' = [0,1)$. Then $\sup_X(X') = 1 \neq 2 = \sup_Y(X')$. □

Notice also that, if a subset X' of a poset X contains $\sup_X(X')$, then the supremum is the greatest element. Similar property holds for the infimum. Following is an important result concerning the set of real numbers.

Lemma 2 (The supremum property) *Every non-empty set of real numbers that is bounded above has a supremum. This supremum is a real number.*

Proof Let $X \subseteq \mathbb{R}$ be a non-empty subset of real numbers, and let $U := \{u \in \mathbb{R} \mid u \geq x \text{ for all } x \in X\}$, the set of upper bounds of X. By assumption, U is non-empty. By the axiom of completeness,[1] there exists a real number α such that

$$x \leq \alpha \leq u, \text{ for all } x \in X \text{ and } u \in U.$$

Because $x \leq \alpha$ for all $x \in X$, α is an upper bound of X. This implies that $\alpha \in U$ with the property that $\alpha \leq u$ for all $u \in U$, and hence, α is a supremum of X. ■

The following definition introduces the notion of a maximal (minimal) element, which is, in general, different from a greatest (least) element.

Definition 14 (Maximal and minimal element) *If x' is in X' and there does not exist any $x'' \in X'$ with $x'' \succ x'$ ($x' \succ x''$), then x' is a maximal (minimal) element of X'.*

A greatest (least) element is a maximal (minimal) element. Thus, it is a stronger notion that maximal (minimal) element. A poset can may have any number of maximal (minimal) element. For example, In the *fence* $a_1 > b_1 < a_2 > b_2 < a_3 > b_3 < \ldots$, all the a_i's are maximal, and all the b_i's are minimal. Consider another example. Let X be the set with at least two elements, and $\mathscr{S} = \{\{x\} \mid x \in X\}$ be the subset of $\mathscr{P}(X)$ consisting of singletons. Take the usual set inclusion ordering, i.e., for any two sets X and Y in $\mathscr{P}(X)$, $X \precsim Y$ if and only if $X \subseteq Y$. This is the discrete poset – no two elements are comparable. Thus, every element $\{x\} \in \mathscr{S}$ is maximal and minimal, and for any x' and x'', neither $\{x'\} \subset \{x''\}$ nor $\{x''\} \subset \{x'\}$. Hence, we can conclude that distinct maximal (minimal) elements are unordered.

Example 5 (Consumer theory) In economics, we relax the axiom of antisymmetry, using preorders instead of partial orders. Let $X \subseteq \mathbb{R}^n_+$. Preferences of a consumer are represented by a preorder $\succsim$ (as in Example 2). The preference relations are never assumed to be antisymmetric. In this context, for any $B \subset X$, we call $x \in B$ a maximal element if $x \succsim y$ for all $y \in B$, and it is interpreted as the consumption bundle that is not dominated in the sense that $x \prec y$ for any $y \in B$. The notion of greatest element for a preference preorder would be that of most preferred choice. That is, some $x \in B$ with $y \in B$ implying that $x \succ y$. An obvious application is to the definition of demand correspondence. An element p of $\mathbb{R}^n_{++}$ is called a price system which maps every consumption bundle x into its market value $p \cdot x \in \mathbb{R}_+$. An income level m is an element of $\mathbb{R}_+$. The budget correspondence $\mathscr{B}$ is a correspondence from $\mathbb{R}^n_{++} \times \mathbb{R}_+$ to X which is given by

$$\mathscr{B}(p, m) = \{x \in X \mid p \cdot x \leq m\}.$$

[1]This axiom says that, if L and H are two non-empty subsets of real numbers with the property that for all $l \in L$ and for all $h \in H$ we have $l \leq h$, then there is a real number α such that $l \leq \alpha \leq h$ for all $l \in L$ and for all $h \in H$.

The demand correspondence maps any price p and any level of income m into the set of maximal elements (with respect to $\succsim$) of $\mathscr{B}(p, m)$, which is given by

$$\mathscr{D}(p, m) = \{x \in X \mid x \text{ is a maximal element of } \mathscr{B}(p, m)\}.$$

It is called demand correspondence because the theory predicts that for given p and m, the rational choice of a consumer x^* will be some element of $\mathscr{D}(p, m)$. □

4 Functions

4.1 Definition and Properties

A function from a set X to a set Y is a rule that assigns to each element x of X a unique element y of Y. We say that y is the image of x under f, and write $y = f(x)$. Conversely, x is an element of the preimage or inverse image of y, written $x \in f^{-1}(y)$. Formally,

Definition 15 (Function) *Let X and Y be two sets. A function f from X to Y, written $f : X \longrightarrow Y$, is a relation from X to Y with the property that for each $x \in X$ there exists a unique element $y \in Y$ such that $(x, y) \in f$.*

The underlying set X is called the *domain* of f, and $f(X)$ is its range. If $A \subseteq X$, then its image under f is given by

$$f(A) := \{y \in Y \mid \text{there exists } x \in A \text{ such that } y = f(x)\} = \cup_{x \in A} \{f(x)\}.$$

Given a subset B of Y, its inverse image is given by

$$f^{-1}(B) := \{x \in X \mid f(x) \in B\}.$$

If $f : X \longrightarrow Y$ and $f(X) = Y$, then f is *surjective* or *onto*. A function f is *injective* or *one-to-one* if for any two distinct elements $x_1, x_2 \in X$, we have $f(x_1) \neq f(x_2)$. A function is *bijective* if it is both one-to-one and onto, i.e., if each element $y \in Y$ has a unique inverse image in X. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$. Then their *composition*, $g \circ f$ is a function from X to Z and is defined by $(g \circ f)(x) = g(f(x))$. We have $(h \circ g) \circ f = h \circ (g \circ f) = h \circ g \circ f$, i.e., the composition is associative.

Theorem 1 *Let $f : X \longrightarrow Y$ be a function and $\mathscr{B} = \{B_i \mid i \in I\}$ a family of subsets of Y. Then*

(a) $f^{-1}(\cup_{i \in I} B_i) = \cup_{i \in I} f^{-1}(B_i)$,

(b) $f^{-1}(\cap_{i \in I} B_i) = \cap_{i \in I} f^{-1}(B_i)$.

Proof The proof is left as an exercise. ■

Definition 16 (Increasing function) *A function $f : X \longrightarrow Y$ is (strictly) increasing if $x' > x''$ in X implies that $f(x')(>) \geq f(x'')$.*

4.2 Representation of Binary Relations

Definition 17 *A function $f : X \longrightarrow \mathbb{R}$ is a utility function, where $X \subseteq \mathbb{R}^m_+$, representing preference relation $\succsim$ if, for all $x, y \in X$, $x \succsim y \iff f(x) \geq f(y)$.*

Proposition 1 *A preference relation $\succsim$ can be represented by a utility function only if it is rational.*

Proof To prove this proposition, we must show that if there is a utility function that represents the preference relation $\succsim$, then $\succsim$ must be complete and transitive. First, because $f(\cdot)$ is a real valued function defined on X, it must be that for any $x, y \in X$, either $f(x) \geq f(y)$ or $f(y) \geq f(x)$. Because $f(\cdot)$ represents $\succsim$, this implies either that $x \succsim y$ or that $y \succsim x$. Hence, $\succsim$ must be complete. Next, Suppose that $x \succsim y$ and $y \succsim z$. Because $f(\cdot)$ represents $\succsim$, it must be that $f(x) \geq f(y)$ and $f(y) \geq f(z)$. Therefore, $f(x) \geq f(z)$, which implies that $x \succsim z$, and hence $\succsim$ is transitive. ■

4.3 Examples and Graphical Representation

Let $f : X \longrightarrow Y$ be a function, and consider the subset of $X \times Y$

$$G_f := \{(x, y) \in X \times Y | y = f(x)\}. \tag{1}$$

The above set is called the graph of the function f. In what follows we consider real valued functions $f : X \longrightarrow \mathbb{R}$, where $X \subseteq \mathbb{R}^2$, that assign to each $x = (x_1, x_2) \in X$ a number $f(x_1, x_2)$ in $\mathbb{R}$.

Example 6 Following are the examples of real valued functions from $\mathbb{R}^2$ to $\mathbb{R}$.

(i) $f(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2$ (Linear),

(ii) $f(x_1, x_2) = A x_1^{\alpha_1} x_2^{\alpha_2}$ (Cobb-Douglas),

(iii) $f(x_1, x_2) = \min\{\alpha_1 x_1, \alpha_2 x_2\}$ (Leontief),

(iv) $f(x_1, x_2) = A\left[\alpha_1 x_1^{-\rho} + \alpha_2 x_2^{-\rho}\right]^{-\frac{1}{\rho}}$ (Constant elasticity of substitution). □

Given a real valued function $f : X \longrightarrow \mathbb{R}$, consider the set $I(x) := \{y \in \mathbb{R}^2 | f(x) = f(y)\}$. Thus, if $x \sim y$ then the function $f(.)$ is constant on the set $I(x)$. Let this value be α. The graph of $I(x)$ defines an indifference curve or a level curve at the level α. The graphical representation of functions from $\mathbb{R}^2$ to $\mathbb{R}$ is often difficult since the graphs are three dimensional. We represent such functions by their level curves. Figure 1 depicts the level curves of the Cobb-Douglas function in Example 6 are as follows. How such level curves are drawn is a topic of discussion of Chapter 3.

5 Correspondences

A *correspondence* φ from X to Y is a function that to each element x in X assigns a non-empty subset $\varphi(x)$ of Y. Hence, a correspondence from X to Y, denoted $\varphi : X \rightarrow\rightarrow Y$, is a function from X to $\mathscr{P}(Y)$. Notice that a function associates with every element $x \in X$ an element $y \in Y$. In saying this we are not making the distinction between the element y in Y and the subset $\{y\}$ of Y which consists of one point only. Given a correspondence $\varphi : X \longrightarrow \mathscr{P}(Y)$, consider the subset G_φ of $X \times Y$ defined by

$$G_\varphi := \{(x, y) \in X \times Y | y \in \varphi(x)\}. \tag{2}$$

The set G_φ is called the *graph* of φ. Conversely, if G is a subset of $X \times Y$ such that for every $x \in X$ the set of points $y \in Y$ with $(x, y) \in G$ is non-empty, then G is the graph of a correspondence, i.e.,

$$\varphi(x) := \{y \in Y | (x, y) \in G\}. \tag{3}$$

We now introduce the notion of inverse image(s) of a correspondence. Let $\varphi : X \to\!\!\to Y$ be a correspondence, where $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, and W be any subset of Y. Define the upper inverse of W under φ, denoted $\varphi_+^{-1}(W)$, by

$$\varphi_+^{-1}(W) := \{x \in X \mid \varphi(x) \subseteq W\},$$

and the lower inverse of W under φ, denoted $\varphi^{-1}(W)$, by

$$\varphi^{-1}(W) := \{x \in X \mid \varphi(x) \cap W \neq \varnothing\}.$$

Example 7 (Best response correspondence) Consider the following two-player normal form game $\Gamma = \langle N, \{S_i\}_{i\in N}, \{u_i(\cdot)\}_{i\in N}\rangle$, where $N = \{1, 2\}$ is the set of players, $S_1 = \{s_1, s_2, s_3\}$ and $S_2 = \{t_1, t_2, t_3, t_4\}$ are the strategy sets of players 1 and 2 respectively, and $u_i(s, t)$ is the payoff function of player i which is given as follows. A strategy t^* of player 2 is a *best response* against a strategy s of player 1

Table 1: Payoff matrix

	t_1	t_2	t_3	t_4
s_1	1, 1	1, 1	0, 0	0, 0
s_2	0, 0	2, 2	2, 2	0, 0
s_3	1, 0	1, 0	0, 0	3, 3

if $u_2(s, t^*) \geq u_2(s, t)$ for all $t \in S_2$. In a similar fashion one can define the best response of player 1. Notice that the best response of player 2 is a correspondence φ as there is no unique optimal strategy against s_1 and s_2. The best response correspondence of player 2 is given by $\varphi(s_1) = \{t_1, t_2\}$, $\varphi(s_2) = \{t_2, t_3\}$, $\varphi(s_3) = \{t_4\}$. In this example, the upper and lower inverses of the set $W = \{t_2, t_3\} \subseteq S_2$ are given by $\varphi_+^{-1}(W) = \{s_2\}$ and $\varphi^{-1}(W) = \{s_1, s_2\}$. □